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ABSTRACT

Math Notebook is a publication issued 10 times a year, with each issue focusing on a particular learning problem in mathematics and its diagnosis and remediation through practical, workable strategies for use by teaches, parents, and tutors. All the articles were written by Mahesh C. Sharma, with the exception of "The Japanese Soroban," by Frances R. Metallo. Topics included are: (1) "What Should We Do: U.S. Pupil Lag in Math Ability"; (2) "Problems in Algebra: Part One (Why Some Students Have Difficulty in Learning Algebra)"; (3) "Problems in Algebra: Part Two: Word Problems Involving Linear Equations"; (4) "Reversal Problems in Mathematics and Their Remediation" (indicating that some early grade students have difficulty in reversal of numerals, letters, and geometrical shapes); (5) "How to Take a Child from Concrete to Abstract"; (6) "Levels of Knowing Mathematics" (including intuitive, concrete, representation, abstract, application, and communication levels); (7) "Division: How to Teach It?"; (8) "The Japanese Soroban: A Tool that Enhances Instruction through Concrete Learning" (dealing with teaching the abacus); (9) "Ontogeny Takes Over Phylogeny: How Do We Learn Arithmetic?"; and (10) "Mathematics Culture." (YP)

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MATH NOTEBOOK

(From Theory to Practice)

Information For Teachers/Parents of Children with Learning Problems in Mathematics

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IN THIS ISSUE

What Should We Do: U.S. Pupil Lag In Math Ability

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- Assistance for the adult student who is returning to college and has anxiety about his/her mathematics ability.

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WHAT SHOULD WE DO: U.S. PUPIL LAG IN MATH ABILITY, SEVERAL STUDIES FOUND

(Mahesh C. Sharma)

Millions of young Americans have trouble with reading, writing, and mathematics. This is not because they are incapable of learning these skills. American students are poorly prepared in the basic knowledge and skills of science and mathematics. Recently people have become aware that American scientific and technical education is in trouble. Engineering jobs are going begging — even in today's economy. The military is worried about the poor technical skills of its recruits. We have a shortage of science and math teachers in our schools. This has been the conclusion of several research studies and by several national commission reports here in the United States. Now new evidence from several international studies ranks the mathematical ability of American primary and high school students "among the lowest of any industrialized country." Data from three major studies made public at the beginning of this year show that the deficiencies are apparent from the first grade through the twelfth and in virtually every aspect of school mathematics from arithmetic to calculus. "When it comes to academic mathematics the United States is an underachieving nation," said Kenneth J. Travers, one of the researchers connected with one of the studies. A new report "The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective," found that American 13-year olds were competitive with students elsewhere in computation and other basic skills but performed "well below" them in such higher tasks as solving word problems. The Americans' algebra achievement was found to be average, but their performance in geometry was "among the bottom 25 percent of all countries." Among high school seniors headed for college, the researchers found that not only were American students "substantially below" the international average but also that the best American students performed only "at or near the average" of students in other countries. "In most countries, all advanced mathematics students take calculus," the report said. "In the U.S., only about one-fifth do." On a fifth-grade geometry test, for example on 12 questions, the American classes correctly answered 3 to 5 on average, as against 10 to 12 in the other three cities: Beijing, China; Taipei, Taiwan; and Sendai, Japan. Similar patterns were found when tests on 12 major areas of mathematics, from arithmetic to problem-solving, were administered to a random sample of 1,200 students. "The only area in which Americans approached the performance of the other countries was in the speed of calculation," an area where American mathematics curriculum has traditionally placed heavy emphasis.

These findings are similar to the conclusions drawn by the National Commission on Excellence in Education, which has given the nation an alarming assessment of the education of our children in the areas of mathematics and science.

According to a government survey, after a decline in the 1970's, American students are doing better at simple arithmetic but still have trouble using those skills to solve problems. We do see some change: younger students are improving in the basics and older students are doing worse in higher level skills. But the picture is not uniform, the 1982 National Assessment for Educational Progress survey found that black and Hispanic students showed greater improvement in basic mathematics skills than white students but that the minority students still trailed significantly behind white students at all ages. Much of the positive changes that occurred according to this assessment can be attributed to improved performance on rather routine items. For example, our children know how to do routine arithmetic, but they are not good at knowing when to do it — or, more accurately, which calculations to use to actually solve a problem involving numbers. In other words, many children are functionally illiterate in mathematics even though they can perform computations fairly well.

This grave deficiency in problem-solving ability seems hard to understand at first. After all, every school mathematics program includes problem solving among its goals, and every math course includes practice on solving various kinds of problems that simulate real-life situations. It is because the teaching of calculation skills is based on an accurate and knowable theory, but we still do not have a well-developed theory of how to teach mathematics problem solving. Much of the teaching of school arithmetic is based on an analysis of arithmetic learning done more than 60 years ago by Edward L. Thorndike, one of the giants in experimental psychology. His theory has been modified and refined over the years, but it has not been fundamentally changed. Most schools are still using the same techniques of teaching mathematics as fifty years ago when the pool of learners was selective. "New Math" was not a real answer to this problem.

Mathematics performance is not the only subject whose preparation is not adequate. Just how bad things have become is illustrated by the fact that all four major users of high school graduates — business, industry, colleges, the military — must conduct remedial courses in mathematics and English. The military spends some $60 million a year in developing basic reading and mathematics skills. Between 25 and 40 percent of enrolled college freshmen require remedial work.

The reasons for this poor performance are many. For the Commission on Excellence in Education the key reason for the poor preparation in mathematics and science is the poor training of teachers and that they are misused on nonacademic tasks during the school time. At colleges where they are trained they are among the very lowest in quality as measured by Graduate Record Examination aptitude test scores. For example, between 1977 and 1980, prospective teachers averaged 440 in verbal and 467 in quantitative test scores, while all Graduate Record Examination takers averaged 497 and 531, respectively. Researchers involved in several such studies and others cite many other reasons also. Compared to few years ago, fewer students in high school are taking mathematics courses beyond the required for graduation. Because of the lack of properly trained mathematics teachers many of the mathematics teaching positions are being filled by teachers who are not properly trained and as a result fewer and fewer mathematics courses are being offered. There is a general lack of options for demanding courses in mathematics. Students want to take those courses in which they have to do less work and they get away with it. Mathematics curricula in American elementary and junior high schools are repetitious and dwell too long on basic arithmetic at the expense of more demanding topics with real applications. It is diffused and lacks coherence. Americans place relatively less importance on mathematics in relation to other subjects, such as reading; the routine practice of placing students in classes by ability condemns many to classes in which low expectations become self-fulfilling prophesies. American students and parents have different attitudes from their counterparts elsewhere, especially in Asia, about what produces success in mathematics. The pool of students in schools is more diverse and needs better and diverse methods of teaching.

Improvement in mathematics education

Curricula

Several reports have said that American elementary school consists of "great deal of repetition and reviews, topics are covered with little intensity." As late as the eighth grade, teaching is still "arithmetic-driven, resembling much more the end of elementary school than the beginning of high school." And many a times this involves meaningless, paper-pencil repetition. By contrast, at this level, "France places a great deal of emphasis on geometry and Japan provides an intense treatment of algebra." A document just produced by the United Nations Educational Social and Cultural Organization shows that Russian students are doing serious algebra and geometry in junior high school. The improvement in math curriculum has to begin in the lower grades: solid foundation in arithmetic and problem solving and meaningful and demanding mathematics beyond that.

Public attitude about mathematics

Another difference lies in the relative importance given to reading than mathematics. In several major international studies, interviews with teachers, parents and students revealed significant differences in attitudes toward education that the researchers said could help explain the achievement differences between American and Asian students since the overall intelligence of the children was determined to be the same. "Chinese and Japanese parents and teachers place more emphasis on mathematics, while mathematics seems to be a weak third in the American triad of reading, writing, and arithmetic." In American elementary schools not even one hour a day is devoted to mathematics compared to 2-4 hours a day to reading and language arts.

"The Japanese and the Chinese believe that people are basically the same and that the difference between success and failure lies in how hard you work," said Harold W. Stevenson, a professor of psychology at the University of Michigan and a leader in this research cited above. "Americans give more importance to native ability, so they have a less incentive to work hard in school." These are two extreme views about human potential. The truth is somewhere in the middle. The Asian immigrant students are proving that by hard work many of the inequities of the environment and individual difference can be overcome. Last year a large percentange of graduating high school valedictorians were Asians. The latest Westinghouse Science Talent Search list shows that the hard work by many of the minority students, particularly the Asians, has resulted in their being on that list in proportionately higher numbers.

Many American students don't see the value of math and science as important in their personal lives. Neither do their parents, teachers, and especially, school counselors inculcate that in them. Unfortunately they are being poorly prepared for life; they are being short changed. They lack "self-organization" and the capacity to commit themselves to anything beyond their egoes and instant gratification.

Parental involvement

American parents, the researchers found, place great emphasis on intellectual stimulation for young children. Ninety-eight percent of American mothers reported reading to their kindergarten children, as against 84 percent of Japanese mothers and 76 percent of Chinese mothers.

As a result, American children achieved "remarkably high scores," well above those of their Chinese and Japanese counterparts, on tests of general information. But they found, once children reach school age, American parents tend to turn education over to the schools. By contrast, Chinese and Japanese parents become heavily involved in their children's schoolwork, encouraging homework and financing tutoring. For the Chinese and Japanese, the school is an important institution but they believe they have part in it has the effect of mobilizing the family.

In views about what leads to academic success, Americans were found to place far more emphasis on native ability as opposed to disciplined study. "Japanese and Chinese students have more motivation to work hard. Homework is not something to hate. It becomes something that enables them to achieve success.

The Americans' stress on natural ability also contributes to problems resulting from sorting students in classes by ability, or "tracking." The Japanese and Chinese schools don't have tracking, so the message to everyone is the same: Work hard. There is general feeling amongst many Americans that they do not have the aptitude for mathematics therefore what is the use of trying. More Americans than any other group are willing to accept and declare that they are poor in mathematics and suffer from mathematics anxiety. Children get a clear message from this declaration.

Methods of teaching

Almost anyone connected with education is advocating changes in American education, particularly mathematics education. The ideas behind the changes are also right — that mathematics should not concentrate on memorization and rote-learning but on understanding concepts and problem-solving. But the key is recruitment and training of talented persons to the profession. The reform in training calls for changes in the content of training but also in the way they are trained. In addition there must be changes in training of school administrators who dominate the curriculum but who have not kept up with changes in science and technology's mathematics needs of the modern society and the advances that have taken place in learning and teaching of content. Few school principals have a good academic background in mathematics or science. This makes it difficult for them to help their teachers develop effective mathematics programs. School administrators, particularly elementary principals, should be master teachers/academic leaders.

The current situation is that elementary teachers know a lot about the child psychology but they are poorly prepared in content, particularly in mathematics and science. The high school teachers, on the other hand, may know enough about their subject area but many of them really do not know how students conceptualize mathematics. They suffer from the so called "college professor" syndrome, they want to teach like college professors. Most college professors are good researchers and are not trained in teaching. If high school teachers have to copy untrained teachers they do not put a high premium on becoming great teachers. There is a need for elementary teachers to know more mathematics content and for the high school and junior high school teachers to have master teachers as role models. They need to know more about the process of teaching and to know how a person in junior and senior high school acquires concepts in mathematics. Both, elementary and high school teachers have to make important changes in teaching in order to make mathematics more interesting. They themselves have to become learners in order to become good teachers.

Children and T.V.

Young people in American society have been captured by TV and the entertainment industry. They have reduced attention spans for serious work, underdeveloped reading and writing skills and poor attitudes towards math and science. Parents will have to curtail the time children watch T.V. Several important research studies have shown that T.V. watching (more than few hours a week) is negatively correlated to achievement.

Homework

Many children must be won over to learning and, most important must be helped to develop learning habits. A child cannot learn if his impulses are not controlled. Homework is a useful tool to teach self-discipline of learning. But homework often becomes ineffective because it isn't carefully monitored and properly evaluated by the teacher and parents do not pay enough attention to it. Then, instead of teaching good working habits, it signals the opposite: independent effort doesn't count. Homework has to be a school level policy with a definite amount of homework assigned daily. Homework has to be assigned carefully. In mathematics, homework must have three components: The first third must be cumulative in nature, second third reflecting the material covered that particular day, and the final third must be either the interaction of the two or an extension of that particular day's work. Ideally, for grades 1-3 it should be 15 minutes, grades 4-5 about 20 minutes, 6-8 about 30 minutes, and in high school about 45 minutes each day.

Role of educators

Many of the problems in education can be traced to "pop psychology" and on the misinterpretations of major learning theories of Piaget and other such prominent researchers. But the major blame should be shared by people like Holt and others who have oversimplified the ideas about learning processes and who think that the child will learn whenever he wants to learn and our aim is just to wait for that to happen. It does happen, but that happens only rarely when the conditions are conducive to learning. In most cases it is not possible. In such cases we are setting up situations just to drift. Parents cannot just make their children's total education as a school's responsibility. They have to change the attitude that learning takes place only during the school hours. Parents and schools will have to find ways of extending this time by working out cooperative responsibilities and devise plans for that to happen. Learning should be continued even after the school time. American children, in the name of all round development, are overscheduled in athletic and extracurricular activities. "Events" of all kinds take precedent over all the academic activities. Many school administrators, because of this pressure from parents, devote inordinate amount of time on non-academic matters.

Exit Requirements

No success is possible without defining and setting standards of accomplishment. There should be definite, enforceable, meaningful criteria for graduation at important junctures of schooling from elementary to high schools.

Conclusion

We need good teachers who are able to instill love for learning; who use methods of teaching that are time tested and based on solid research in learning and teaching. We need classrooms where patterns of good teaching are visible: teacher teaches a concept, practices it with children, helps students apply the concept and skill, assesses children's learning, reteaches if necessary, pays close attention to children's diverse needs and their learning personalities, reinforces the related skills on a regular basis, and enriches to keep the interest of children. We need schools where administrators are not just managers but are academic leaders and master teachers. The key to all this are parents who are partners in children's learning and who realize the importance of learning beyond school walls. And finally a society which respects learning and respects those who provide opportunities for learning.

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PROBLEMS IN ALGEBRA: PART ONE

(WHY SOME STUDENTS HAVE DIFFICULTY IN LEARNING ALGEBRA)

Editor: Mahesh C. Sharma
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PROBLEMS IN ALGEBRA: PART ONE

(WHY SOME STUDENTS HAVE DIFFICULTY IN LEARNING ALGEBRA)

Mahesh C. Sharma

I. INTRODUCTION

There are many students who do fairly well up to the sixth grade in mathematics, but some of them as they approach algebraic concepts in the eighth grade and even some of the prealgebraic concepts, begin to experience difficulty in conceptualizing and using mathematics ideas. For many of them, this is the first time that they experience the trauma of failure in mathematics. This failure has serious effects on their attitude toward mathematics. As a result, their interest in mathematics begins to decline and in some cases, they even begin to intensely dislike mathematics. In a few other cases this same phenomenon may occur later, for example, in ninth grade mathematics and then for still others again in the first year of college mathematics.

In our highly technological age, everybody no doubt needs some background in mathematics. Mathematics permeates the whole society and its use seems to assume ever increasing importance as our society becomes more technological and complex. Mathematical methods and thinking are not the prerogative of scientists, engineers and technologists only, their use in analyzing individual behavior, to study opinions and attitudes is also increasing. Is it not ironic, then, that the subject which has assumed such prominence in society is also that which is most closely related to failure in school and after?

Recent research studies have suggested that it is very likely that the structure of classroom interaction creates ability differences among students which grow during the years of schooling. In searching for causes of increasing differences in mathematical aptitude, perhaps the simplest explanation rests on the assumption that such differences are due to predispositions to mathematical thinking, with the further implication that nothing can be done really to change the situation. The understanding of elementary mathematics in the first years of primary school is based on precondition such as the acquisition of notions of conservation of quantity which are, in their turn, embedded in exploratory activity outside the school. The genesis of general mathematical abilities is still little understood. The possibility that extra-school experience with mathematical or premathematical ideas influences school learning cannot be excluded. Classroom interaction in fact produces growing differences in mathematical aptitude and achievement by a system of positive feedback mechanism which increase high achievement and further decrease low achievement.

Brigid Sewall's report on the use of mathematics by adults in daily life shows that

> Many of the people interviewed during this inquiry were inhibited about using mathematics; this led them to avoid it as much as possible and in some instances it has affected their careers. This inhibition was most marked among women who had specialized in humanities. The more educated were affected to a much greater extent than the less educated.

Unfortunately almost all of these high school, college students and adults have studied mathematics (in the form of arithmetic) for several years during elementary and junior high school years and yet at the end of so many years of mathematical experience they still have many gaps in their ability to use mathematics skills, even arithmetic, in very simple ways. Algebraic concepts and their applications pose even more difficult situations. The problem is two fold: one that of learning mathematics and the other of applying mathematics to mathematical and non-mathematical situations.

II. APPLICATIONS OF MATHEMATICS

The understanding of most mathematics lies in its applications. This is particularly true today as high school and college students have to take more and more mathematics. The demands for mastery of mathematical concepts is increasing as a whole range of intellectual disciplines are using more and more mathematical (quantitative) models to understand and solve problems in a diversity of fields. The most common and popular mathematical tools and models used in different fields are from algebra. Almost everybody needs some proficiency in mathematics to function effectively in today's technological society. Many high school graduates, today, are restricted in career choices by the lack of adequate and proficient knowledge in mathematics, particularly algebra.

Mathematical modeling is a critical component of applications of mathematics to other disciplines as can be seen in recent textbooks in other subject areas. Clearly most recently published mathematics textbooks and several commission reports, have increasingly emphasized application. For example, the National Council of Teachers of Mathematics (NCTM's) **Agenda for Action** emphasizes applications; its focus on problem solving clearly indicates the importance of modeling. The study of mathematics is no longer restricted to students in engineering and the physical sciences. From business and management to agriculture, from medical professionals to social sciences, students are required to take mathematics that at least involves algebra and more often they have to go further than algebra many times even to calculus.

IIA. ALGEBRAIC APPLICATIONS

One key component needed for entry into meaningful and rewarding jobs especially into scientific and technological professions is fluency in algebraic concepts and their applications. The major forms in which we see algebraic applications is forming and solving algebraic equations. The first step in this process involves translating data into mathematical models (in this case algebraic equations), working with the mathematical model(s) developed (solving the algebraic equations), and then interpreting the mathematical solution into the original problem form and terminology.

III. DIFFICULTIES IN LEARNING ALGEBRA

People have difficulty in learning algebra for many reasons. Some of them are: socio-cultural expectations (as in the case of many girls who were not encouraged to learn higher mathematics), psychological (cognitive deficiencies and psychological blocks in learning mathematics becuase of early failures in mathematics. Recurrent failure in anything creates anxiety about that task. This is particularly so if it is found that the person has to learn it and has to master it to be successful in his field of endeavor and this is definitely true in the case of learning mathematics), lack of understanding about the utility of algebraic concepts, nature of abstraction in mathematics, poor teaching of mathematics, etc.

Many of the factors cited above are related with the individual, but we must not neglect the inherent difficulty of the subject of mathematics itself, and algebra in particular. The mathematics curriculum, from elementary school to college increases in abstraction, symbolism and obscurity. As students progress from year to year in mathematics, the letters they use, the concepts they learn, become increasingly abstract and to them they become ambiguous. This high level of abstraction and the extensive use of symbolism keeps many students away from mathematics and those who attempt it, for many of them it is a difficult and challenging task.

In this issue of the **Math Notebook**, we describe how college and high school students solve or fail to solve algebraic equations and the analysis of the types of errors that students make in the process of solving linear algebraic equations. In this respect, we will, first, try to identify and classify the difficulties students have in learning algebra in general and solving linear equations in particular, and then to study the mechanisms that might have produced those difficulties. This includes the study of reasons such as: whether the mechanism that produced those difficulties was lack of a specific piece of knowledge or the failure to carry out a process. As a result of this we will suggest counter measures for these errors. For example, an error that results from a specific wrong idea cannot be overcome by emphasis on careful or neat work, while an error in execution might be. Second, we will give examples of successful and unsuccessful execution of solving linear equations abilities so that one can compare the behaviors that these two kinds of students exhibit in their solutions. This will provide us with differences in the way good problem solvers work on the algebraic word problems involving linear equations that go beyond the presence or absence of errors. Third, we will identify the preskills needed and learned by the student to successfully conceptualize algebraic concepts and solving algebraic equations.

Under the category of algebraic equations we intentionally limit our topic to linear algebraic equations. This is because linear equations involving one or two variables are the first algebraic models students apply in their algebra courses. Once a student has mastered these equations, he/she is generally prepared for other equations with minimum of difficulty. In the future issues of this publication we plan to carry the inquiry into other types of equations. We focus upon the strategies they use, the behaviors they exhibit, and the skills they need to solve these problems. We will also include the descriptions of some of the cognitive functions that are called upon for successful mastery of the collection of skills. We call these as collection of skills because solving equations is not a simple task, as a matter of fact it is quite complex. It requires a combination of mathematical, cognitive, and thinking skills. Solving algebraic equations involves a complex interplay of many forms of knowledge: both mathematical and non-mathematical.

However, in many instances it is not necessary to have a high level of proficiency in arithmetic before a student could be introduced to algebra. This is particularly so if the student has a high level of thinking (cognitive) skills. Of course, if the proficiency in arithmetic is present it is definitely helpful. We have found that proficiency in algebra is not directly related to proficiency in arithmetic. As a matter of fact, at the high school level, in many cases, it is advisable to begin algebraic work with a student who has done poorly in arithmetic and who has seen arithmetic work so many times that he no longer has any motivation for doing arithmetic. This is particularly so in the case of slow learners and learning disabled students in mathematics. Most of these students, who have difficulty in arithmetic, every year in September, they are introduced to the same arithmetic (beginning with operations with whole numbers, fractions, and related concepts) and by November they are so disinterested in arithmetic that they are not motivated to do any work in regular arithmetic. This ensues the same cycle: the cycle of seeing unmotivating material, of lack of interest in this material, failure, frustration and avoidance of mathematics. Since most of these students have a low threshold for failure as soon as they see some failure in mathematics the problem begins. This story is repeated year after year for many students.

Of course, there are several reasons for this cyclic phenomenon. And the lack of variety both in instructional materials and teaching styles are definitely two of them. Poor teaching, lack of concrete materials, and absence of interesting teaching materials and models are amongst the several reasons for these students to lose interest in mathematics. The lack of importance of the subject matter on the part of these students is another reason for their failure in mathematics. It is difficult for these students to understand the use of any meaningful mathematics content in their day to day life. As a result we have found that it is useful to begin simple algebraic work with these students and then digress to arithmetic as we need it in the study of algebra. We, at the Center for Teaching/Learning of Mathematics, have introduced many students to algebra without first achieving a high level of proficiency in arithmetic. And as a result they show a renewed interest in arithmetic which was absent before. The important thing is to give them some success in simple algebra and then integrate the other mathematics material into the introductory algebraic material.

IV. SOURCES OF ERRORS

Many people complain of suffering from mathematics anxiety. Mathematics anxiety in earlier grades is just a symptom of a student's having difficulty in mathematics and it reflects the failures which they have had in elementary mathematics, but when this anxiety persists during teenage years and beyond, particularly during high school and in college years, it may become one of the causes of mathematical learning difficulties. The person has mathematics anxiety and therefore he/she avoids mathematical experiences and as a results does not do well on mathematics tasks. Therefore, the person does not attend to any new mathematics material. This results in more anxiety about mathematics. And this cycle repeats itself. The external manifestations of errors in their mathematics work and the resultant anxiety must be identified and eradicated before this anxiety is internalized and becomes a causative factor for the student's learning problems in mathematics. Once identified and causes known, the success in mathematics is a possibility. Without proper identification of the nature and causes of the problem, remediation is a much more difficult task. The presence of mathematics anxiety is not necessarily a sign of inferior intelligence. There are many persons who are highly successful in other fields of endeavors but are anxious about mathematics. In many cases the mathematics anxiety is the outcome of early failures in mathematics which could have been due to many factors: Early introduction to difficult abstract concepts, poor preparation for those concepts, and poor teaching are definitely among some of the major factors. Many math anxious students do not even want to attempt algebra because they feel that failures in arithmetic will jeopardize their ability to learn algebra.

Students facing difficulty in algebra are not just those who have done poorly in elementary school years or those who are intellectually or cognitively lower. Some students who are intellectually quite superior also face difficulty in algebra if it is presented earlier than they are ready for it. For example, in many schools, superior mathematics students from seventh grade are given the opportunity to study first-year algebra in the eighth grade. This course is generally presented as the first course of an accelerated mathematics program that culminates with advanced placement calculus in the senior year of high school. Unfortunately, first-year algebra proves to be a stumbling block for some of the students who enroll in the accelerated mathematics program. Unsuccessful students either drop algebra courses and revert back to regular eighth-grade mathematics or complete the course only to have to repeat it in the ninth grade. In either case, intelligent, previously high achieving students experience the trauma of failure.

Although the difficulties of these students are quite different than the students we were talking about earlier it is quite interesting that it is not that the students who have had difficulties in arithmetic concepts also have difficulty in learning algebra. In most cases the students of the type discussed in this paragraph are misplacements in these early algebra courses. Once the problem of placement have taken care of these students begin to do all right in algebra. In the later part of this article we will be talking about the specific problem of these students.

In our experience, we have found that there are three groups of students who do well on prealgebraic and arithmetic concepts. The first group is one with individuals exhibiting high level of memorizing abilities but poor reasoning skills. They are able to memorize formulas, facts, procedures and 'recipes' of mathematics operations easily. Let us call the students with this characteristic as group A. The second group of students exhibit a high level of reasoning abilities but may not have high memorizing abilities or may not feel inclined to memorize mathematics material. They conceptualize mathematics material quite well. Instead of memorizing formulas, facts, and procedures, they "figure out" the formulas when they need them. They understand the general principle of the concept but may not pay attention to the subtleties and nuances of the concepts. They also do quite well in arithmetic and prealgebra, but their responses are not that swift. Since they "figure out" the formulas and problems almost from scratch they are always racing against time and, therefore, may not complete questions on the examinations and tests. Let us call the students with this characteristic as group B. The third group, we will call group C. They have the characteristics of both groups A & B. They are the best students. They are quick in reasoning and retain most of the information needed to solve problems. Then there are students who do well in mathematics by sheer hard work because they are highly motivated to do well.

Let us follow three representative students from these groups in the accelerated algebra program in the eighth grade. We have observed, that quite a good number of the students from the first group begin to have difficulty in the algebra course around the beginning of November. The reason being that up to this point, the material in the course is generally a review of prealgebra work. The teacher is reviewing basic facts and concepts relating to rational numbers and from about the beginning of November abstractions, extrapolations, and generalizations of ideas begin to take place and they might be difficult for these students. The enrollment of these students in the accelerated algebra course is a misplacement. If these students are placed in a strong prealgebra program they will do well in algebra after a year.

The students from the group B, do quite well up to February or March, but some of them begin to have difficulty when in a problem several concepts are being combined, cumulatively there are lots of formulas are being used in the solution of problems, and the procedures have become quite elaborate. For example, they might have done quite well on factoring of quadratic trinomials, but when teacher begins the rational fractions or solving equations involving quadratic expressions they begin to have some difficulty. In their lack of paying attention to "minute and subtle" parts of the concepts they have a cumulative aspect of missing several things by the time they reach this point. And those things become quite important when one is applying algebraic concepts. However, these students are not misplacements in the algebra course, they need a little more time to process the subtleties and nuances of the ideas that they missed the first time around. When we begin to provide them a little more time on a weekly basis and help them process the minute parts of the concepts they begin to do well.

Of course, the students from the third group, C, do not have any difficulty unless they are unmotivated or some other factors arise. They finish the algebra course with good marks.

As a result of these observations, we feel that enrollment in the accelerated algebra courses should be limited to those students who have demonstrated high performance (85% or higher) on previous mathematics course and should have completed concrete operational stage (third stage of cognitive development according to Piaget) or exhibit the presence of abstract thinking according to some cognitive test.

There are several reasons for students' difficulty in algebra courses, particularly in an introductory course, whether they are in junior or senior high school or even college. Some of the important ones are: **the**

abstract nature of algebraic operations; faulty processing and then incorrect extensions of arithmetical algorithms to algebra; learning inefficient procedures and failure to make the transition from arithmetic to algebraic operations; difficulty in making abstractions, generalizations; and **inability in detecting patterns and relationships between arithmetical ideas and algebraic concepts.** The presence of any of these or a combination of them could create difficulty in mathematical operations at the higher mathematics level.

The student's initial problems with mathematics and the manifest errors must be identified and eradicated before they begin to interfere with his mathematics learning processes. It is important for the teacher to identify the type of errors by observing the student's work and interviewing him about his mathematical work. Because, too often, the correct answer on a paper-pencil test may be achieved through inefficient and incorrect procedures which are difficult for the teacher to detect in a regular classroom setting. Formally structured and unstructured interviews with the student provide a method for determining a student's understanding of algebraic concepts and finding out the logic that the student might be using for solving the algebraic problems.

V. NATURE OF ALGEBRA

Algebra may be defined as the symbolic language of real numbers; understanding algebra is knowing the relationships among the language elements and where to use these relationships to solve problems; and doing algebra is manipulating the symbols to produce other relationships and solutions to examples, exercises, and problems. Any disability in reading and writing, symbolic manipulation, reasoning, ability to abstract, and relating arithmetic facts to algebraic concepts may cause difficulty in doing well in algebra.

Algebra at the high school and college level is not only generalized arithmetic but also a means of modeling practical situations. But the modeling process is far more complex than is generally considered. To understand algebra, it is essential to see this generalization process and the extensions of arithmetic into algebra. Good students are able to see these relationships and are able to generalize them to abstractions. For mathematicians this generalization — from arithmetic to algebra is quite obvious. Good students can easily see that the facts and operations that they saw in arithmetic lend themselves to generalizations and result in algebraic formulations. But amongst the students with mathematics learning problems very few of them are able to share this perspective. Since they do not have a grasp of the interralationships between different concepts in arithmetic they have difficulty in seeing the generalizations and abstractions in the form of algebraic concepts. Therefore, one has to go back and forth from algebra to arithmetic and from arithmetic to algebra. The teacher must relate the two for the students for example:

(1) (a)

$$\begin{array}{r} 35 \\ \times\, 42 \\ \hline 70 \\ 140 \\ \hline 1470 \end{array}$$

(b)

$$\begin{array}{r} a + b \\ \times\, (a + b) \\ \hline + ab + b^2 \\ a^2 + ab\phantom{{}+b^2} \\ \hline a^2 + 2ab + b^2 \end{array}$$

Here the teacher show the similarities between the two problems.

(2) 918756 $3x^2 + 16x + 8x + 4$

are polynomial expressions, first in powers of ten, and second as powers of x.

Algebra is a collection of many powerful concepts but the one that permeates all of algebra and has the most versatile applications in and outside the field of mathematics is forming equations from given real life problems and solving them. This is one of the topics that creates a great deal of difficulty for many students and becomes a hurdle to learning algebra.

VI. ALGEBRAIC EQUATIONS

Although forming equations and inequalities is not seen by the students for the first time when they reach an algebra course, it begins to take place even in the lower elementary school mathematics program. The only thing is that in many instances not enough emphasis is placed by teachers on this topic at that time. In an elementary mathematics program, enough attention should be given to open sentences, such as $5 \times 4 = n$, $5 \times n = 30$, and $54 - 6 = n$. One obvious reason for presenting such sentences is to provide computational practice. Another important role of an open mathematical sentence is to provide a mathematical model for a class of different physical problems. But more importantly, the understanding of these open sentences is the beginning of abstraction. Yet another reason is the importance of writing equations involving variables representing real world situations. In both the computational and modeling roles the ability to solve open sentences is very important. As a matter of fact this open sentence activity is the beginning of seeing the mathematical facts in the form of equations — a good preparation for algebra. There are obviously many instances where a teacher can bring symbolic form of mathematics in the arithmetic classes so as to prepare the students for algebra. Conversely, an algebra teacher can use arithmetic examples of the following type to set up patterns, for demonstrating algebraic ideas and abstractions. For example, while introducing students to equivalent fractions teachers may give examples as:

1/2 = 2/4 = 3/6 = ... 10/ ? = 15/ ? = ?/ 14 = ...

As in these examples, most teachers stop at this point with using numerical values only in presenting the fraction expressions. It will be of utmost importance and value if the teacher continued the expressions to the forms:

1/2 = 2/4 = 10/? = ?/14 = /? = glob/? = A/? = F/? = ...

In other words, if the students are introduced to non-numerical forms and values in expressions where traditionally only numerical values are used, they will begin to see that variables are not something they should be afraid of. Algebraic expressions and equations then begin to look familiar.

When students see the connections between algebra and arithmetic they begin to see the generalized notion of algebra. Many times I have heard students expressing how much pleasure thev derive on understanding algebra, when I have shown the connection betwen arithmetic and algebra. Their anxiety about mathematics also diminishes considerably as a result of seeing these connections.

VII. THE PROBLEM OF THE "MISSING ADDEND"

The understanding of the place holder positions, operations, systematic property of equality, existence of a whole number solution, and the factor of number size are all related to performance in solving open sentences and therefore a precursor to algebraic equations. Observations have shown that performance on algebraic equations increased significantly when the students were shown the simple equations using place holders in arithmetic concepts first and then were introduced to algebraic equations. The addition and subtraction sentences showing arithmetic facts in placeholder forms help the students in understanding the rudimentary forms of algebraic equations. The following example illustrates how placeholder equations can be introduced at the elementary school level.

One of the earliest examples of open sentence is an equation where one of the addends is missing. For example:

$$2 + \square = 6.$$

Written in this form of this equation, it involves three concepts, addition, unknown quantity, and the equality. As a result it is a secondary concept. (**Secondary concepts** are those concepts that are combinations of several **primary concepts.** Primary concepts are those concepts that are independent, autonomous, and simple. For example, addition, subtraction, multiplication, and division of single digit numbers, adding 3x and 5x, single step operations, etc. are good examples of primary concepts. Secondary concepts, on the other hand, are combinations of two or more primary concepts. For example, adding and subtracting with regrouping, adding 5x, 3y, 7x, and 6y, etc. are examples of secondary concepts). In the example considered above let us first discuss the meaning of the verb 'equal' and its customary symbol '=' with respect to adding or subtracting. Many children have trouble with symbols; for instance:

$$3 + 4 = 7 \text{ or } 3 \times 4 = 12$$

are familiar and manageable, while

$$7 = 3 + 4 \text{ or } 12 = 3 \times 4$$

strike them as untrue or strange. The understanding of the symmetric property of equality or the commutative property of addition or multiplication are significant factors in children's performance on sentences/equations. For example, children understand the forms:

$$a \circ b = c \; (a \times b = c, \; a - b = c)$$

better than sentences of the form

$$c = a \circ b \; (c = a \times b, \; c = a - b).$$

Because of familiarity with the form $a \circ b = c$ and teachers' repeated presentation of arithmetic facts in the form of $a \circ b = c$ rather than presenting them in other forms, such as $c = a \circ b$, children have facility with the first form of equations. As a result when many of the students see equations of the form

$$7 = 4 + x$$

they consider this to be more difficult than the equation of the form

$$x + 4 = 7$$

because of the unfamiliarity of the first form. For example

$$(2/3)\,x = 9/5$$

seems to them easier than

$$9/5 = (2/3)\,x$$

although, mathematically both are exactly the same. The underlying reason for this difficulty is lack of reversibility of thought a cognitive skill. To decrease this impression on the part of the students it is important that the teachers impress upon the idea that both are equivalent by presenting both types of equations in the introductory lessons. This should be a recurring theme throughout the mathematical curriculum and the idea must be specially impressed upon in the elementary school mathematics curriculum.

This difficulty is evident even in the case of algebra students. Many students, for example, are able to identify the slope of a line and its intercept easily if the equation of the line is in the form ($y = mx + b$), e.g., if the equation is $y = 3x + 4$. If the same equation is expressed in the form $4 + 3x = y$, many of the students from the above group will have difficulty in identifying the slope ($=3$) and the y-intercept ($+4$). Although, mathematically both equations are equivalent. But, with lack of experience and since the equation is not in the form that is familiar to the student, he has difficulty with this equation. Most teachers in their presentation of this material present the equation in y = m x + b form (although in exercise sets students see other similar expressions of the problems), the students are familiar with the standard format. Therefore, this form is relatively easy for the student.

Many children probably see the '=' sign as a command to perform the operation indicated at the left-hand side of the equation, rather than a passive statement of a fact — which it is. If it were a command, then there could be an infinite number of 'answers' such as 7 = 5 + 2, 6 + 1, or 9 - 2, etc. These equations are examples at the arithmetic level, but they have lingering effects on learning algebra. Concrete examples and appropriate discussions of the reversibility of the operations, ideas, and equations are important aids in understanding algebraic equations.

VIII. TYPES OF ERRORS INVOLVED IN SOLVING ALGEBRAIC EQUATIONS

In our work, at the **Center for Teaching/Learning of Mathematics,** we find that students having difficulties in mathematics at the seventh through ninth grade levels (ages 11 through 14 approximately) by far outnumber the other groups with similar difficulties. There are several reasons for this, but the most important one is that in most schools algebra or pre-algebra becomes the core content of mathematics programs at these levels. Many of those children who were fairly comfortable with the arithmetical operations find algebraic concepts difficult to understand and they find algebraic operations more abstract and devoid of concreteness which was the nature of arithmetic. Arithmetic and prealgebra did not involve complex abstractions, generalizations, and extrapolations. The algorithms are long, complex, and involving several operations and ideas. In arithmetic the list of procedures to be recalled in the situation of solving a problem was a small one. But, now in the case of algebra they have to recall a particular procedure from several likely choices. For example, to determine which method out of several methods (solving an quadratic trinomial by factoring, completion of a square, applying quadratic formula, or by observation) is most effective, is a difficult one. Many students can solve a problem if the procedure is straight forward and no choices are to be made.

In the literature on the study of students having difficulty in learning algebra and algebraic equations, faulty algorithms, inefficient procedures and failure to make the transition from mental arithmetic to algebraic operations (Davis & Cooney, 1978; Driscoll, 1982; Hart, 1980; Herscovics, 1979) are the majority of reasons cited for difficulty in solving equations. These studies claim that most times these factors remain undetected by the teachers as a result remediation does not result in success. In a classroom with many students, a teacher, most times, only has the opportunity to see the written work and therefore does not have the chance of seeing the thinking of the student in solving a problem. Researchers point out (Kruteskii, 1976) that students' beginning "external" errors must be identified and eradicated before they become "internal, mental operations" which are far less accessible to remediation. In **Psychology of Equation Solving: An Information Processing Study**, Carry, Lewis, and Bernard (no date) argue that the errors that they observed in the study can be divided into three types: operator, applicability, and execution. The three types have different preventive or remedial measures.

> Operator errors seem to reflect incorrect knowledge, or incomplete knowledge that is overextended under the pressure of solving the problem. The procedural steps generally are applied erroneously and seem distorted and fragmented, but seldom completely unrecognizable. They generally are images of correct operations. Many of these students combine fragments of different procedures and develop it into a new procedure. If they get the answer to even one problem right, they are convinced of the applicability of their invented procedure. The task of preventing such errors seems to be divided into three parts: making initial learning more successful, discouraging the later **construction or reconstruction of incorrect operators,** and correcting existing incorrect operators.

This is not possible unless the teacher is willing to take the detailed analysis of the student's work in order to identify the evidence of such a situation. The same detailed analysis of initial errors is important in improving initial learning. The initial teaching has to be careful, descriptive, and very structured. For each possible operation, generalization, abstraction, and procedure the teacher should provide several examples, counterexamples and situations where the given procedure and operators work and the places where they fail to be applied. It seems students are inclined to take a rather wholistic view of an operator, therefore they overextend the applicability of the operator. A detailed step-by-step analysis of the task resulting from the particular operator must be taken. The process of forming examples and counterexamples is an important step in initial learning and therefore in teaching.

In operator errors, with respect to **deletion** and **recombination** errors in particular it seems students are inclined to take a rather generalized view of the operations they are learning, and with the nature of the likely over generalization in mind it might be possible to choose examples in teaching that would bring out the needed discriminations. This behavior is particularly evident in the case of qualitatively oriented mathematics personality students. (See Sharma, 1986)

Applicability errors involve mishandling of the order of operations, particularly parentheses. Here students assign false groupings, wrong distributions, wrong collection of variable terms. Even accurate equation solvers show this common error reflecting a real misunderstanding of these operations. Again task analysis and presentation of counterexamples and the effects of wrong operations and right operations must be identified for the students.

Execution errors generally occur when a student is mistakenly generalizing an arithmetic operation or a student is taking shortcuts. As a result misexecution and partial executions occur. Once again these errors are quite common among even accurate equation solvers. Good solvers in order to achieve speed and efficiency commit these errors.

Memory retrieval difficulties also play a great role in students having difficulty in solving algebraic equations. The three things that help a great deal in alleviating these problems are: first, asking the student to give the script of the procedure before he/she solves the problem; doing at least 10 to 15 minutes of oral mathematics in class each day; and planning daily homework. We realize that homework is part of every teacher's teaching approach, but we are asking for a special type of homework. According to our experience we recommend that the homework should have three components — 1/3 of the homework should be cumulative in nature (it should include representative examples from everything you have done up to that point), another 1/3 of the homework should contain problems exactly like the ones you have introduced in the classroom that particular day, and final 1/3 of the component of homework should be designed either to extend the ideas discussed in the class or should involve the interaction between the other 2 one-thirds of the homework.

Nature of Errors

Research has identified several types of errors in solving linear equations. Most representative errors could be categorized are:

1. **Arithmetic**
   - basic facts
   - defective algorithm
   - wrong operation
   - wrong order of operation

2. **Properties of Numbers**
   - associative
   - distributive (sign error in distribution)
   - commutative
   - operational/ruled based

3. **Procedural**
   - misuse of the property of equality
   - + property of =
   - x property of =

coefficient error +
coefficient error x
wrong operation (wrong/inverse operation)

4. **Conceptual**

   order of operations (opposite sign)
   like terms (conjoining) (combining unlike terms, combining different variables) partial distributive
   zero annexation
   misunderstanding constants as variables
   others

5. **Mechanical/Perceptual**

   careless/random
   incomplete operations

Some mathematical difficulties are student specific. For example, the quantitively oriented students are able to apply the procedures correctly but they may not be able to determine the correctness of the solution or the nature of the answer beforehand. Whereas the qualitatively oriented students do see the answer intuitively but commit errors in the procedural aspects in solving equations. They make mistakes in signs and order of operations. These students when presented with a procedure see it in its generalized form and therefore have difficulty in seeing specific situations where it does not work. Therefore, it is necessary to show several examples where the method works and the situations where the method may not be applicable. A second attack Carry et. el suggest on the problem of initial learning is through metaknowledge of right or wrong. If the student can distinguish a correct operator from an incorrect one, then he or she can edit out the wrong guesses and generalizations that must inevitably flow from limited examples and ill-understood explanations. We have found that when students are given examples to see the applicability or inapplicability of a given procedure then they begin to discriminate. For example, when a student is learning to factor the difference of squares $(x^2 - a^2) = (x - a)(x + a)$ it is necessary not only to show in which conditions we apply this method of factoring but also to show under which conditions this will not be applicable. For example, in this situation it is necessary to show that it will not work in the following illustrative situations: $x^2 + a^2$; $x^2 - 2$; etc. One of the important ways of learning a concept is to ask students to give counter example for that particular concept. When a student is looking for a counter example he analyzes the definition and the conditions of the concept more closely and carefully and therefore learns the concept better. Let us take an example of applying this procedure to solving algebraic word problems.

Many students are not able to discriminate the situations where a particular procedure is applicable. There are two ways the teacher can develop this ability of discrimination in students. First, a procedure must be developed inductively (developing a pattern) so that the student can derive the procedure themselves under the supervision of the teacher (and some help). For example, let us say we are working with problems such as:

1. A Balab weighs 3/4 lbs. more than a Coroc. Together they weigh 6 lbs. What are their weights?
2. A dog weighs 3/4 as much as a cat. Together they weigh 28 lbs. What are their weights?

The following presents an example of the questioning process that can and develops the understanding of the pattern that are needed to see the word problem translation process.

Teacher: A balab weighs 10 lbs. more than a coroc and the coroc weighs 20 lbs. What is the weight of a balab?

Student: 30 lbs.

Teacher: Yes. What did you do to find the answer?

Student: I added 20 and 10 to give me 30.

Teacher: Write it please.

Student: 30.

Teacher: Yes. You are right. (Teacher writes 20 + 10).

Teacher: A balab weighs 11 lbs. more than a coroc and coroc's weight is 57 lbs. What is the weight of a balab.

Student: 68 lbs.

Teacher: Yes, What did you do?

Student: I added.

Teacher: Yes. (Teachers writes 57 + 11.)

(Teacher should ask several more questions using numerical values. Then write the answers as a pattern:

Balab	Coroc
20 + 10	20
57 + 11	57
. . . .	. .
. . . .	. .

student's attention should be brought to this pattern.)

Teacher: Express in the same way as before. A balab weighs a lbs. more than a coroc, and the coroc weighs b lbs.

Student: b + a.

(If a student has difficulty, then help him see the pattern by identifying b as 20, 57, ... and a as 10, 11, ... At this point, the teacher may have to give few more numerical examples to show that pattern.)

Teacher: A balab weighs 3/4 of a pound more than a coroc. Express it in the same way. What do you need to know to find the weight of a balab?

Student: The weight of a coroc.

Teacher: Do you know the weight of a coroc?

Student: No.

Teacher: What do you do (What do you call it) if you do not know its value?

Student: x.

Teacher: What is the weight of a balab if it weighs 3/4th of a pound more than a coroc? And you know that the weight of a coroc is x lbs.

Student: $x + 3/4$.

(The teacher should do several examples to make sure that the student grasps the pattern and the logic behind this pattern.)

Second important step in developing the discrimination is to show a counter example where the procedure just learned will not work. For example, now the teacher should discuss the problem:

> A dog weighs 3/4 as much as the cat. Together they weigh 28 lbs. What are their weights?

The teacher should discuss this problem and show the similarities and differences between these two problems. If we do several examples without a counter example of the type discussed above a student could easily might persevere in that procedure.

In discouraging the construction or reconstruction of erroneous operations knowledge of right or wrong is still clearly important, since the problem of filtering correct operations from wrong guesses is fundamentally the same in construction as it is in initial learning. (Carry et al, 198?; Matz, 1979)

Most of the applicability errors involve mishandling the order of operations particularly the parentheses with the terms in an expression being assigned a false grouping. These errors are quite common even amongst accurate solvers, so it is unlikely that they reflect a real misunderstanding of the syntax of expressions. Rather, it seems that the patterns of parentheses is just not as salient as it should be, and when solvers are blocking out the structure of expressions it may be overridden by such other grouping factors as the repeated form in equations. Unfortunately, it seems that it is the absence of parentheses, rather than the presence, that is most frequently overlooked, so devices that draw attention to the parentheses might be ineffective. Students must get practice in writing expressions that express the same idea mathematically so they begin to see what can be done when the parentheses are there and when the parentheses are not there.

In upper elementary and junior high schools, some teachers use mnemonic devices to help students master the order of operations, the same or similar kinds of mnemonic devices are useful for high school and college students. For example, the acronym PEMDAS is extremely useful. Here the letters stand, for in the order of operation: **P**-parentheses, **E**-exponents, **M**-multiplication, **D**-division, **A**-addition, and **S**-subtraction. In other words, first simplify the expressions in the parentheses (this applies to brackets and braces also), then perform exponential operations, then multiplication and division (whichever comes first in the expressions), and finally subtraction and addition (again in the order of occurrence). In simplifying one uses the acronym CLT (combining like terms).

Example: $5 - 9\,(7 - 8x + 3) + 1 = 9.$

Many students even at high school and college level are likely to perform the operation $5 - 9 = -4$, which is an error. But, when they are asked to apply PEMDAS, the errors of this type are drastically reduced, rather quickly.

Many of the difficulties that arise in algebra can sometimes be traced to the trends described in the previous sections in arithmetic and in general these difficulties can be classified into three major areas: difficulties of instruction; difficulties of conceptualization; and curricular difficulties.

1. Instructional Difficulties

a. Arithmetic difficulties: Many students cannot transfer skills from the arithmetic concepts to algebra; from algebraic equations to word problems; and from mathematics classes to science classes. For example, a student can solve

$$2x^2 + 5x - 3 = 0 - 4x.$$

for x in his mathematics class but cannot solve

$$s = 5t^2 + 2t + 6$$

for t in his physics class.

b. Students often lack proficiency in the mathematical skills that they know; for example: Calculate T in the following

$$T = 2p\,(100\text{cm}/980\text{cm}/\text{sec}^2)\ 1/2$$

c. Students usually are reluctant to make approximations; for example: In finding the value of d satisfying the following equation.

$$d/16 + 6\,(2d/1100) - d/121000 = 9.60 \text{ (d in feet)}$$

How many students would try to calculate an approximate value of d rather than an accurate one. They will try to solve it for the given numbers and will be lost in calculations."

The students who are procedurely oriented, they show the inability of lack of approximation more often than others. Many a times teachers neither introduce it nor emphasize this as an important skill.

d. Students usually have difficulty with the mathematization of a

> situation. Too often, students learn a collection of procedures in the form of "recipes" without understanding the ideas and concepts behind them which they apply without understanding and which they quickly forget. Many times the correct answer on a paper-pencil test may be achieved through different and incorrect procedures which are difficult for the teacher to detect. In such a situation the teachers have to look for methods that can provide information for determining a student's understanding of algebraic concepts and allow probing into the rules used by students in making particular errors.

Work toward easing the above difficulties must be the responsibility of each individual teacher. However, a mathematics teacher cannot always provide explicitly for transfer of skills to a great number of fields. The teacher in the elementary school can provide anchor points for bridges from arithmetic to algebra and similarly the algebra teacher can provide similar relationships from algebra to arithmetic and algebra to other fields by using a variety of symbols other than x and y. This lack of teaching transferability from arithmetic to algebra and from algebra to science and other technical subjects is the effect of improper teacher training. Teachers are exposed to a very limited amount of material in subjects other than their field.

Examples in the first category might include:

$$1/2 = 2/4 = ?/10 = 24/? = 8/? = a/2a = b/? = (\alpha/? = \frac{\triangle}{?} = .$$

Whereas examples in the second category would be slightly difficult, but if the algebra teacher consults with teachers of other subjects, particularly science teachers and textbooks in these subjects he/she can come up with good number of examples. The training of mathematics teachers should include some familiarity with other disciplines where mathematics is used.

2. Conceptual Difficulties

This type of difficulties may arise due to inappropriate concepts required to be learned by students who are not yet ready for those concepts. The curriculum may require a topic that is conceptually very difficult at the particular age considered. For example, Gubrud & Novak (quoted in Walton, 1975, p.11) showed that most 16 year olds were able to incorporate material on vector addition into cognitive structure with clarity and stability but that most 14 year olds could not do so in the same learning period. Perhaps conceptual problems are at the heart of the often reported difficulties with several of the other algebraic concepts. The idea of understanding what the roots of an equation mean geometrically or the relationship between the discriminant and the nature of roots of a quadratic equation are some examples to show that many of the algebraic concepts are not particularly understood by 14 and 15 year olds. They can manipulate the operations but not the concept. The idea of understanding what the roots of an equation mean geometrically or the relationship between the discriminant and the nature of the roots of a quadratic equation falls in this category of concepts. Similarly the concept of linear inequalities is a difficult one for young students in an algebra course.

3. Curricular Difficulties

When you look at different types of equations you find an approximate hierarchy in the difficulty of algebraic equations. Problems of the type:

1 A $8 (x + 5) = 5 (x + 4)$;
1 B $5 (x - 4) + 2 (5 + 2x) = 2x - 12$;
1 C $5x + 2 (x - 1) = 14$

were the easiest. But, if the equation 1 C is changed to the equation

1 D $5 + 2 (x - 1) = 24$,
1 E $5 - 2 (x - 1) = 24$, or
1 F $5x - 4 (x - 2) = 25$,

we find that many students make mistakes in solving 1D - 1F, even if they can solve 1C.

The difficulty seems to be with the order of operation in 1 D, difficulty with the distribution of -2 is the problem in 1 E, and the distribution of -4 in 1 E. In the example in 1 E the problem is only in the distribution part whereas in 1 B some students show difficulty in both the order of operation and also in distribution. Thus, the order of operation and the distribution both contribute to the difficulty in these problems for some students.

IX. REMEDIATION OF DIFFICULTIES IN SOLVING ALGEBRAIC EQUATIONS

The nature of remediation has to be planned according to the level of the student. Students at the high school level have the most difficulty in this area. The reasons for this are twofold: first, they do not see the explicit reasons for doing well in mathematics and they also do not see the practicality of learning algebra; second, you cannot effectively discuss the learning process and the nature of their difficulty in solving algebraic problems. As a result one has to find circuitous ways to deal with the problem. They neither have the understanding of the psychological processes underlying the errors, nor are they clearly able to see the causative relationships between the thinking processes and errors. In our clinical experiences, we have found a few entering high school students who were able to see these relationships. With these students the discussions of their problems, their possible causes and the strategies of remediation is always helpful. Even in the case of those who have difficulty in understanding the psychological processes underlying the difficulty, it is still advisable to discuss these as they still benefit from the knowledge that remediation is possible. Since the students are not there voluntarily it is the teachers' task to invent methods of motivating them to learn how to solve algebraic equations.

Whereas in the case of students at the postsecondary level the remedial activities could be somewhat easier. There are advantages and disadvantages in remedial activities with students at the postsecondary level. The advantages far outweigh the disadvantages. For the most part, the students participate in the learning activities voluntarily and can at their own initiative engage in the learning activity. It is easier to discuss the nature of their learning problem and causes of their difficulty. Therefore, it is easier to remediate the problem as their fullest participation is possible. On the negative side, adult learners bring a wide range of prior experiences to their learning. These encompass not only experiences with content of mathematics which result in both cognitive and affective learning but also experiences that include inefficient methods of solving problems and faulty procedures and perceptions. For example, it is very difficult to engage math anxious adults in a mathematics learning experience. It is the affective contexts that pose the maximum resistant to remediation of their learning problems.

The math anxiety experienced by many adults can be categorized as: specific math anxiety and global math anxiety. A person with specific math anxiety performs poorly only in the area of math, in other academic areas s/he does quite well. For example, a high school student will score in the 95th percentile on the English component of the Scholastic Achievement Test (SAT) and 20% on the mathematics component. In such cases by providing success at some level of mathematics and then slowly increasing the complexity of mathematics, one can improve the student's performance in mathematics and thereby alleviate the specific math anxiety. The procedure is not that simple in the case of a person with global math anxiety. The person with global math anxiety does poorly in mathematics and also in those areas where he/she is asked to be specific rather than general in their response to questions. This person's self-concept, because of continuous failures in mathematics, has been affected negatively (much more so than the specific math anxiety case.) Generally this person has an anxious personality to start with, has a lower threshold for failure, and has a very negative attitude toward mathematics and mathematics related activities. To help such a person one has to provide not only successful experiences in mathematics but should also provide counseling help to regain his/her self-concept. Without this counseling the student just transfers his/her anxiety from one course in mathematics to another. For example, when the student, with remedial intervention, does well in arithmetic, he says, "I will definitely fail in algebra." When you help him succeed in algebra, he says, "I will fail in geometry." And this transfer continues. Therefore, it is necessary for the teacher to provide both.

Research in teaching algebra is very limited and whatever research is available deals only with identifying behaviors that students exhibit in solving algebraic equations. Very few researchers and educators have devoted any time in providing remedial suggestions for dealing with these identified problems. For example, Brown and his associates (Brown, Burton, et al 1975) collected and interpreted errors from college students in remedial courses. Matz (1979a, 1979b) has also been pursuing the research into the study of mechanisms underlying students' errors while solving algebraic equations. Davis and Cooney (1977) have collected errors from high school students. The theoretical modeling work of Bundy (1975) in developing an equation solving computer program has also thrown light on the problem of solving equations. His model suggests ways of understanding the processes that underlie in solving algebraic equations and the model can be used to compare the actual process used by a student in solving an algebraic equation to the theoretical model. Similarly Neves (1979) is also developing a model of the process of learning to solve equations from examples. But all of this research does not provide any direct remedial suggestions.

After doing content analysis of algebra textbooks, school and college curricula, and observing students learning algebra, many researchers indicated that there are several tasks involved in solving algebraic equations. Koe (1984), for example, through content analysis of ten current first year algebra texts has identified: **four algebraic concepts** (expressions, equality, variable, and equations); **four vocabulary concepts** (like terms, coefficients, inverses, and identities); and **twelve equation types.** In literature four levels of numeric difficulty, four levels of structural difficulty, the placement of the unknown in relation to the equality and the type of numerical values used in the equations are identified as the factors that further created difficulty for the student. Another important factor that creates difficulty for the students is the number of steps involved in solving the problem and whether they have a concrete model in their mind representing the algebraic equation.

According to Koe the development of the complexity of algebraic equations is hierarchical. One can write a sequence of the concepts by keeping in mind the order of difficulty of the concepts according to field properties and the number of steps required to solve the equations.

To find ways of helping students improve their abilities to solve algebraic equations, we need to look at the following factors. **First,** to identify and classify the nature of difficulties students have and if possible to discover the mechanism that produced those difficulties, that is to see whether a particular mechanism was a lack of a specific piece of knowledge or the failure to carry out a process. This knowledge will help a remedial specialist in suggesting remedial and preventing measures. For example, an error that results from a specific wrong idea cannot be overcome by emphasis on careful or neat work, while an error in execution might be. Similarly, if the error is in the execution of arithmetical operation then emphasizing the algebraic conceptual components is not going to help.

Second, we need to observe the behaviors of successful and unsuccessful algebraic problem solvers. This involves both the attitude of students toward algebraic equations and the strategies that differentiate students in these two categories. For example, if the students' difficulty is the fear of failure in mathematics or lack of efficient strategies then that will differentiate the two situations from the point of view of that remedial strategy the teacher adopts for helping these students.

Third, and actually the most important factor is what prerequisite skills are needed to be successful in solving algebraic equations. What kinds of activities can be of help to those students who generally have difficulty in solving algebraic equations. What kind of prealgebraic skill preparation might be helpful before we venture them into algebra or during when they are learning algebra. This also means that we look at how arithmetic is taught in elementary school so that we can prevent the possible problems in the future that the students may have in learning algebra. The teaching of arithmetic generally affects the students' preparation in learning algebra and other higher order mathematics concepts. For example, the failure in arithmetic affects the attitude a student may have towards algebra and other higher mathematics. Should the emphasis, when teaching arithmetic include certain ways of presenting arithmetic concepts? For example, when "new mathematics" was added to the school mathematics content, it was hoped that by emphasizing the structure of mathematics in teaching will help students to prepare them better for algebra and other higher order mathematics. It was hoped that by emphasizing the connections between different components of arithmetic they will be able to abstract ideas and therefore will be better prepared for high school and college mathematics.

An awareness of the structure of a discipline is a powerful aid in learning that discipline. Probably the most important thing for the algebra student (whether he be a 8th, 9th, or a freshman in college) is the fact that the understanding of structure of mathematics places the manipulative skills and facts of arithmetic on a meaningful basis and gives coherence to topics which might otherwise appear unrelated. The understanding of the structure of mathematics also provides a framework for new mathematics concepts so that those ideas can be related and new ones can be created. This is what was expected, when "New Math" was introduced into the mathematics curricula. It was hoped that the teachers would emphasize the structure of mathematics in their teaching practices. It did not often happen adequately because most elementary and junior high school teachers either did not receive enough training in mathematics and mathematics teaching or the teachers did not see the underlying structures of mathematics themselves. Many of these teachers did not really understand and appreciate the objectives of "New Math." As a result many of them misunderstood the whole purpose of the curricular reforms in mathematics teaching and learning. One of the negative outcomes of this misunderstanding was that they even abandoned some of the important practices and techniques that were quite effective prior to this change. For example, teachers thought that the "New Math" curriculum approach told them not to put any emphasis on any kind of memorization even of facts and procedures. Therefore, the structure of mathematics that was hoped for to be learned by students was not achieved and the mastery of arithmetic facts was also affected. There are ninth graders who have not memorized arithmetic facts and that affects their attitude and work in algebra.

Fourth, the research on information processing is also very relevant to understanding the needs of the students who are learning algebra, particularly the learning-disabled.

Wittrock (1979) investigated how learners process information that is presented during instruction. Unless students can be helped in making connections between the information presented in the classroom and past experiences they may not be able to do it themselves. Only through activities in the classroom will students begin to spontaneously generate relevant connections between information presented and past experiences. Average and above average students, ordinarily, are able to spontaneously generate relevant relationships or learn to construct relevant relationships. And most of the time learning disabled students are also able to do so. However, if the material is not explained clearly, they are unlikely to generate relevant relationships. This situation presents a real challenge for a teacher of algebra to make verbal presentations so clear that learners will be able to develop mental images of the rules and relationships of algebra, along with a speaking vocabulary. Ultimately, they must understand the rules and be able to apply them.

Every mathematical idea and therefore algebraic concepts, particularly algebraic equations, have three components: linguistic, conceptual, and skill. These three components require three different types of teaching strategies. The linguistic component should be taught by similar methods that we use in teaching a language. Moreover the language of algebra and its understanding must be emphasized more in the introductory part of the teaching of an algebraic concept. At that point teaching mathematics is almost like teaching a language. The teaching of the conceptual component requires the teacher to present the mathematical idea through models, examples and even experimentation with concrete manipulatives. This is particularly important for students of algebra who have not yet entered the abstract operational stage of cognitive development. This is the part which is severely missing in high school and college teaching. The third component — skill is best taught by coaching. The teacher has to observe the students solving problems and should provide corrective help right then and there. Most of this has to be done orally, so that the teacher can observe the student's thinking processes and provide immediate feedback. This observation and the provision of providing immediate feedback coupled with the opportunity of providing the model for understanding a particular concept makes coaching possible and the learning of that skill sure.

To be successful in solving algebraic equations, one has to know a set of correct operators and steps in order to be efficient and accurate. They also have to know what to do to an equation in order to move it closer to a solution, and what operators will help in that process. To decide what to do to an equation, the solver must know what features of an equation, such as the number of instances of the unknown in a denominator what kinds of numbers are involved in the equation are important in deciding what to do. Thus, to be an accurate solver, the student must have proficiency in the knowledge of appropriate operators and the strategies. The knowledge of operators is dependent on the knowledge of previous relevant subject matter and new concepts being learned. But, operations used (operators) are not simple units of knowledge. Most often they are secondary concepts. For example, the operator like "factoring" in the case of linear equations or quadratic equations involves the understaning of signed numbers (integers) and distribution of multiplication over addition. Similarly collecting like terms (CLT) is dependent on operations with signed numbers. Even operators that do not seem to have other operators as parts seem to have structure, in that a student may have only partial knowledge of them.

Strategy, on the other hand, is a function of experience, efficiency, and economy in effort and time. The strategic knowledge needed to solve algebraic equations is also not simple. For example, a solver needs to distinguish between linear from quadratic equations, and have a different plan for each. When the unknown occurs in the denominator, a check for extraneous roots must be appended to the normal plan. The solver must know that if there are no occurrences of the unknown something has gone wrong, and that multiple occurrences of the unknown must be reduced to one before the equation is solved. Even a simple situation like when the numbers involved in the linear equation are rational or the linear equation has a number in the denominator, the student has to determine what to do with those fractions. At a more tactical level, the solver has to know what operators to collect in several occurrences of the variable, including the situation in which one occurrence has an implicit "1" as its coefficient. Another important area of tactics is the handling of x in the denominator especially the often baffling form "1/x = expression." Many solvers have explicit knowledge of all of these things, many have implicit knowledge, but many have no knowledge of some of them.

Strategy Analysis

1. Skilled solvers may develop more powerful operators, that can produce solutions in fewer steps.

2. One key feature of the skill in solving equations is to be able to see the patterns involved in the equation. The pattern-matching ability of skilled solvers may be better developed, so that the skilled solver is able to see the usefulness of operators that would be passed over by a less skilled solver. The skilled solvers use more powerful operators and as a result the solutions have fewer written steps. The number of written steps is also influenced by other factors, such as the number of steps performed mentally. Such an action would also be expected to shorten the solutions by skilled solvers. Some unskilled solvers also try to shorten the steps which many a times result in errors. The strategy that we use in this situation is to first help the student to give all the detailed steps involved in solving the equation and then ask him/her to see if they can reduce the number of steps in solving the equation or to do several of the steps in their "head."

3. Good solvers have a tighter strategic control: they are constantly aware of the fact whether there is an agreement between their execution of the strategy and the idealized strategy. As a result they have a greater use of repeated subexpressions as units of analysis. Formation about operators, strategies, and preskills from arithmetic are not enough. There are other things that differentiate a good solver from a poor one. These are skills that a student acquires by practice and through analysis of his work. When a teacher provides analysis of and feedback for student's work, the student begins to differentiate between good operators and poor operators, and wrong answers and right answers. Improvement in these skills takes place more effectively when teachers devote more time on oral problems and their analysis before a student attempts written problems.

There are two major ways that researchers have suggested so that students might test the validity of their ideas, use of principles and checking. (Carry, et al, 198?)

"Principles" refers to the basic mathematical properties of the arithmetic operations that underlie the algebraic operators. For example, the fact that multiplication distributes over addition, that multiplication and division are inverses, and so on. A student who knows these generalized principles well could reject operations that could not be derived or generalized from them. Unfortunately, the basic principles are almost as numerous and complex as the algebraic operators based on them, so students will be doubtful about them. Further, the testing of an operation by use of the principles is an exercise in proof, and may require creativity. Teachers need to connect the algebraic operators with the underlying arithmetic principles. For example, teachers need to show that cross multiplication is related to the basic principles of addition of fraction and to the concept of ratio and proportion. Without this understanding students apply it in a routine manner.

Checking one's work is an important exercise in understanding the process. It involves reversibility of thought and the ability of transforming from one system to another. The question a student has to ask constantly is "Can I get back to the previous step by "undoing" the operator I have just applied?" If the answer is in the affirmative, the result obtained by the application of that operator is right. Of course, it does not ascertain that whether that particular operator is applicable in that particular situation or not. That requires mastery of principles and strategies. Carry and his colleagues divide checking into two distinct methods: global checking and local checking methods. Local methods are those that can provide information about the correctness of a single step. The checklist is a potentially useful diagnostic tool in identifying the types of errors which students make when solving linear equations in one variable. The use of "think aloud" interviews provides insight into the thought processes used by students in solving equations. The "think aloud" technique also allows exploration of a student's understanding of algebraic concepts. Of these, trial evaluation seems to be a better method to prefer, for poor solvers, because of its generality and ability to detect both stable and transient errors.

While the basic trial execution method is applicable directly only to steps that replace an expression by an equivalent one (reductions in the terminology of Matz, 1979), it can be extended to most deductions, steps that transform the entire equation. Most deductions can be analyzed as performances of the same operations on both sides of an equation, and simplification. Under this decompositions, it is the simplifications where many errors occur, and the simplifications are reductions, that can be checked by trial evaluation.

Despite some limitations, it appears that "checking by trial" teaching might help students filter good ideas from bad. Lack of knowledge of how to check is probably not the only reason students do not check. Checking is taught as a means of increasing accuracy. As long as checking is just a way to be more sure of the answer to a given problem, students are free to use it or not according to the accuracy and time pressures they feel. Good solvers, however, need to see checking as a way of evaluating their knowledge, not their answers only.

Another method that might be of help is asking students to tell a story about the equation. By demonstrating the equation in a concrete manner or through a story will elicit better information regarding the student's understanding of the meaning of an equation.

Students need to know how to distinguish incorrect operators from correct ones but they also need to see how this ability fits into the learning task. They need to know that they will form misconceptions and make errors, despite their best efforts, and that there are specific actions they can take to deal with these difficulties. Testing and correction of knowledge should be seen as a normal and important activity, not as an option, or worse, as something good students could avoid. Discussions of common errors, with emphasis on the often sensible analysis that lies behind them, might help develop the necessary perspective on learning.

(to be continued in next issue)

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PROBLEMS IN ALGEBRA: PART TWO:

WORD PROBLEMS INVOLVING LINEAR EQUATIONS

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WORD PROBLEMS INVOLVING LINEAR EQUATIONS

(Continued from the previous issue)

Mahesh C. Sharma

As one goes about shaping a mathematical approach to thinking, it is very important that the students be encouraged to and made to see that applying mathematics to the solution of word problems is mostly a step-by-step process. The use of mathematics for the solution of any "real world" problem breaks down into three stages. The **first** of these consists in translating the "problem situation" into formal mathematical model of the situation i.e., to a clearly stated mathematical problem. This is the stage of formalization. This is the most difficult step for many of the students as most of them have experiences in solving "word problems" with limited and predetermined parameters.

The solution of the mathematical problem (most times it is a well defined algebraic equation) thus formulated by the methods developed in mathematics constitutes the **second** stage i.e., that of solving the given problem within the given mathematical model. This stage requires the knowledge of the mathematics involved in that equation. If the algebraic equation involved is a standard one. This step is the easiest one. The **third stage** is that of interpreting the solution of the mathematical problem and the application of that solution to the initial situation and of comparing it with the initial situation.

The first and the third stages of this process are based on the student's ability to translate from one language system to another. Here it involves translation from English language to mathematical language and vice-a-versa.

Most teachers and teaching methodologies pay extensive attention to the second stage of the problem solving process: Solving formulated problems. The first and second stages, the key stages in the application of mathematics to problem solving are based on the possibility of describing phenomena and processes in mathematical language, i.e., on the use of mathematical symbols: formulae, diagrams, mathematical expressions, and drawings. It is all language. To be good in these two stages calls for proficiency in mathematical language. The language of mathematics is the language of science and like any language it must be capable of clearly expressing scientific or social ideas and facts. That is why students must not only follow the clearly and logically presented oral and written material, but also devote time and pay due attention to the proper use of the language of mathematics in the presentation of mathematics material and solving problems. This practice is the only way to acquire the ability to translate from English language to mathematics and vice-a-versa.

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I. NOTATION, SYMBOL AND TRANSLATION

There has always been considerable interest in the relationship of reading and mathematics word problem solving. Research has generally been correlational in nature: Reading affects mathematics word problem solving. It applies not only to word problems involving arithmetics concepts, but also word problems involving algebra. In addition to the reading difficulty contributing to word problem 'problem', a large number of students do not understand the use of letters (symbols and variables) in algebra, particularly in algebraic equations. Misconceptions that students have surrounding the use of letters in equations contribute significantly to the difficulty. (Rosnick, 1981) Students tend to view the use of letters in equations as labels that refer to concrete entities. Many of these students think that the variable is just a label for some thing rather that it may represent a number. For example, P stands for professor or professors not the more abstract 'number of professors.' As a result they have a great deal of difficulty with translations. The results of extensive research (Rosnick, 1981; Clement, Lockhead, and Monk, 1981, Clement & Kaput, 1979, Clement, 1980; and Rosnick, 1981) based on diagnostic tests, video tape interviews, and the microanalysis of those interviews — indicate that students (including for the most part, first year engineering students and/or students who have taken one or two semesters of calculus) have a very difficult time with translation from English to algebraic forms.

Many students have difficulty in seeing that a variable may represent several things. For example, Davis (1975), in analyzing through clinical interviews found that students have erroneous and/or unstable conception of variable. According to him students are not able to recognize that a variable such as X, in fact represents a number rather than a label to represent just the variable in a problem. Some times it is both.

II. PROBLEM WITH THE SYNTAX OF MATHEMATICS

Some students, such as the one who confuses "one over two x" and "one-half x", or a student who displays confusion with the meaning of x/4 (he does not know if this means "x divided by 4" or 4 divided by x") have trouble with the syntax of mathematical expressions. Many students have difficulty in understanding like terms. This causes problems in solving equations particularly where either two variables or two numeric terms are present. Probably more confusions about syntax are masked by the generic deletion and recombination operators, which blur many distinctions. Since mathematics expressions are representations of calculations, and the syntax is intended to capture the necessary information of the order of operations and assignment of arguments to arithmetic operations, students might benefit from exercises in which complex calculations were to be written as expressions. Writing programs for a computer or a calculator might be good experience in representations of calculations as translations to expression form. Structured interviews provide a method for determining a student's understanding of algebraic concepts and allow probing into the rules used by students in making particular errors. These errors are relatively common even among accurate solvers.

III. THE TRANSLATION PROCESS

In all real life problems that can be modeled into mathematical forms, the key skill needed is the translation from the language or semi-language form of the problem to mathematical form of the problem. This is a skill that can be acquired. There are numerous places in high school and college mathematics curriculum where translation from English expressions to mathematics and from mathematics to English plays a large role in solving world problems. Therefore, one would naturally expect translation and general reading ability to be an important variable for solving word problems in algebra. Research (Phillips, Uprichard, & Blair, 1983) indicates that general reading ability correlates higher with the ability to solve word problems in algebra than general mathematics ability. It is, therefore, obvious that language is a critical variable in perceiving the problem situation. "It may also be that language is a necessary ingredient to perform most of the cognitive processes needed to solve problems (perception, coding, memory, planning, analysis, and transformation). But many teachers because of students' difficulty with word problems deemphasize such problems. Many teachers even skip over the word problems (at least the difficult ones.)

Many of the high school and college teachers generally assume that their students could translate between English and algebra expressions, at least in simple situations. But on close examination (Clement, Lockhead, and Monk, 1981), it is found that students have great difficulty in translation (even those who are taking calculus and other higher mathematics courses.) The reason for this difficulty is that students are rarely asked to construct a word problem or to interpret one in a significant way. When they are solving word problems they are usually given a formula or asked to select the appropriate formula from well-defined (and very short) list and then to manipulate it using techniques from algebra.

Another possibility in their difficulty in not able to solve algebra problems is due to their "overextended application of the representational system of natural language to the formalism of algebra based on the fairly obvious fact that the representational systems of natural language and algebra share several symbols, e.g., '=', and symbol types, e.g., numerals, but do not share either underlying rules of reference or syntax." (Kaput & Sims-Knight, 1983) As a result students apply literal translation processes, called semantic translations by several authors. In semantic translations one matches word for word in the order given. In

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some categories of word problems semantic translation works, but in others it does not work. Many students, even to understand mathematical terms apply the semantic translation process. For example, when students are asked to give the 'least common multiple' of 4 and 6, there are many of them who will give the answer as: 2. The reason for this wrong answer is their processing the term 'least common multiple' from least to common and then to multiple rather than multiple to common and then to least. In essence, they are applying literal translation process to the mathematical term. The same thing happens when they are dealing with other terms such as: greatest common factor, simultaneous linear equations, etc. When they see $(9x + 3) \div (7x + 1)$, they think the operation is

$$9x + 3\overline{)7x + 1}.$$

They even read this as $9x + 3$ divided by $7x + 1$, but the operation they are performing is faulty. This type of difficulty stems from student's misunderstanding of the syntax of the mathematical language.

Problems involving application of algebra require students to translate from one language symbol system to another. In some tasks, students need to translate an English sentence to an algebraic equation and then to interpret the solution they need to translate from algebra to English. In others, students need to interpret information in tabular, graphic or pictorial form into an algebraic equation. Rosnick and Clement (1980) asked their students at the University of Massachusetts "to interpret information in tabular, graphic, or pictorial form into an algebraic equation. The results of their studies indicate that many students fare poorly at these translation activities."

IV. THE REVERSAL

Clement, Lockhead and Monk (1981) identified **reversal of variables** in algebraic equation during translation from English to mathematics as one of the major errors in a student's inability in solving algebraic word problems. Of all the freshmen undergraduate students who could not answer the "Professor-student" type of problem (typically over 50 percent), the majority of them committed reversal errors. "The reversal error," according to these researchers, "appears to be due not simply to carelessness but rather a self-generated stable, and persistent misconception concerning the meaning of variables and equations." They have used specific problems to test this translation ability. The following two examples are typical of such problems:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (P-S problem)
>
> Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four peo-

ple who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered. (C-S problem)

In a group of 150 first-year engineering majors, only 63% were able to answer the "students and professors" problem correctly, and only 27% answered the "cheesecake" problem correctly (Kaput and Clement, 1980; and Clement, Lockhead, and Monk, 1979). It is to be remembered here that these students have studied mathematics beyond simple arithmetic for at least four years, if not more. In other words the problem is not that of lack of exposure to such problems, but misconception about the nature of variables used in the problems and the lack of understanding of the translation processes. There was a very strong pattern in the errors on these problems: two thirds of the errors in both cases took the form of reversed equation, such as $6S = P$ or $4C = 5S$. Further results indicate that students in the social sciences do considerably worse on these questions, as would be expected.

Initially, it was thought that these mistakes were simply careless misinterpretations due to specific wording of the problem. However, the reversal is also quite common in problems that call for translations from pictures to equations or data tables to equations. This suggests that the reversal error is not primarily due to the specific wording used in a word problem. Although, we agree with the findings of this research that reversal is a major problem. We have found through observations of students' work and their comments that they make reversal errors more frequently when the problem involves ratio and proportions concepts as in the case of "S-P" and "C-S" problems. Their translation errors are reduced when the algebra word problems do not involve ratio and proportions concepts.

According to Kaput & Sims-Knight (1983), students who are able to solve the word problems in algebra use an active, procedural approach as against the static-depictive and syntactic-translation approach that unsuccessful students take in solving the algebraic word problems.

Semantic understandings of the word problems is not enough. For example, in the "professor-student" problem, Rosnick and Clement found that many students demonstrated a general semantic understanding of the problem (i.e., that there are, in fact, more students than professors), yet they persisted in writing reversed equations. Their interviews reveal that students have real difficulty in conceptualizing the relationship of variables in equations. According to them, a student may have the "figurative concept" of equation; i.e., that since there are more students than professors, the coefficient, 6, by virtue of the fact it is bigger than 1, should be associated with the "bigger variable," S. This results in the reversed equation $6S = P$. In other words, they comprehended the English language content correctly, but still did not translate the word problem correctly. This demonstrates the fact that translation ability is a higher level ability and therefore, reading ability alone is not enough for solving algebraic word problems. Of course, with arithmetic world problems the reading ability has a greater positive effect. This is the reason that many students with high level of reading ability do better on arithmetic word problems but may not do that well on algebraic word problems.

Rosnick and Clement's analysis of this error identifies two distinct sources for the students' tendency to reverse variables. First is the faulty approach, which they call "word order matching," and described as "Syntactic translation " by Paige and Simon (1966). This is a literal, direct mapping of words from the English language involved in the word problem into the symbols of algebra. In the early grades students translate literally (and sometimes they are even encouraged to do so), for example: When students encounter a simple problem such as:

93 take away 49.

They translate it as 93 - 49. This is a clear-cut example of literal translation. In many of the students' work a similar literal translation is taking place when they make the translation:

There are 6 times as many students as professors, as

$$6 \times S = P.$$

Here the direct mapping (literal translation) has led to a reversal problem. Students are called upon to perform literal translation more often. As the following example illustrates:

> The price of a watch is twice as much as a ring. Together they cost $150. Find their costs?

In this problem, the student directly translates and gets the equation:

Price of a ring	X	dollars
Price of a watch	2X	dollars

$$2X + X = 150.$$

Since students practice with this type of problems more often, they get habituated with direct word mapping type of translations.

V. Double Translation Process

The examples cited above and our clinical experience support the idea that problem solving, particularly word problems, is largely dependent on the development of mathematics language. The question, therefore, is how to teach the language of mathematics. The method that we have found to be very helpful, at least on a clinical level, is to teach mathematics language specifically: its terminology, its syntax, and the translation process. The translation process involves two way translation: The "double translation" process. The double translation involves translating from English expressions to mathematical expressions and from mathematical expressions to the English language. Following are examples depicting the double translation process.

Mathematics Expressions ------------------ >	English Expressions
1. $7X + 8 < Y$.	Eight more than seven times the amount owed to Bob by John is at least as great as the amount he owes me. (X = the amount John owes to Bob, Y = the amount John owes to me.)
2. $2X + 2Y = 48$.	Twice the length and twice the width of a rectangle is equal to forty eight. Or twice John's age added to twice of Mary's age is equal to forty eight.

English Statement -------------------- >	Mathematics Expression
1. The sum of two numbers is 3 times their difference.	$(X + Y) = 3(X - Y)$.
2. The weight of a Balab is equal to 3.4 times the sum of its age and the number of its siblings.	$W = 3.4(A + S)$. (Where A = its age, S = no. of its siblings)

Many teachers have reported to us that when they have provided their students the opportunity of performing this kind of double translation regularly in their mathematics program then their students understand the translation process in algebra problems better and their performance on word problems including algebraic equations improved. Students' understanding of the interplay between mathematics language and English language also improved. We have found the double translation process to be very helpful in our remedial work with children at the Center for Teaching/Learning for mathematics. Therefore, we feel that it is important that the teachers emphasize the double translation process: English to mathematics and mathematics to English. The examples provided above can serve as samples for developing more examples for practicing the double translation process.

We believe that students should be encouraged to gain experience in the translation from English to mathematics and vice-a-versa from the very beginning of the school experience. Existing curricula should be revised to pay much more attention to the teaching and learning of the language of mathematics. In such a situation, the mathematics teacher, in addition, becomes a teacher of the language needed to learn mathematical concepts and skills.

Kaput & Sims-Knight (1983) also make a strong statement about the translations from one system to another in saying, "the abilities involved in translating across representational systems are not only critical to success in applying mathematics to the world of wider experience, but are intimately connected to our natural linguistic and imagistic representational system whose developmental roots date from early childhood and are thus extremely potent in determining our ability to use more formal systems of mathematics."

Although the success in translation from English to mathematics and vice-a-versa is better if this kind of activity begins in elementary school. But it is not too late to start even at the high school level. The outlook is not that bleak. There are certain things teachers can do to improve a student's ability in solving word problems.

Algebraic word problems involving first degree equations in one or two variables, that most students encounter in introductory algebra courses, can be described as direct relation problems and proportional relation problems. In direct relation problems the semantic translation method is an effective method of translation from English to algebra. Example:

(1) The watch costs twice as much as a ring. Both together cost $150. What is the cost of each item?

This problem is translated into the following equation

$$2X + X = 150$$

This is an example of literal translation process.

The "student-professor" and "cheesecake" problems are ratio or proportion relation problems. When students treat proportional relation problems as direct relation problems, they get into trouble. Teachers need to emphasize the difference in these two types of problems and also the differences in the solution process.

According to our observations and with interviews with students who reverse the variables in the problems mentioned above is that the students who reverse the variables in the S-P problem, do not see their reversals. And the reason that they do not see their reversals is that they really have not conceptually mastered the ratio and proportion concepts. Therefore, the reversal problem is the outcome of the interaction of two things. First, students think of variables as names and labels and second, they do not see the actual proportional relationship.

The following activities have been found to be helpful in realizing the difference between literal translation problems and proportion and ratio problems also remediating the difficulty in seeing the proportional relationship.

(A) Translating mathematical expressions into word problems

a) $\square X = \triangle Y$ $X > Y$

What numbers are possible in places of $\square$ and $\triangle$?

b) $\square X = \triangle Y$ $X < Y$

What numbers are possible in places of □ and △ ?

c) $\square X = \triangle Y$ $X = Y$

What numbers are possible in places of □ and △ ?

The first step in remediation is to help students distinguish whether a given problem is an example of literal translation problem or a ratio problem. The analysis of the statements of such problems should help them in doing so. Prepare dozens of problems of both types, arrange them randomly and then ask students to differentiate the language patterns in the two types of problems. After this analysis begin the translation process. The following steps might be helpful.

(B) If it is a literal translation problem, then identify the "characters" involved in the problem and write them vertically:

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and then translate literally to identify the character in terms of variable (variables) and then develop an equation (s). And, then solve the equation.

Example: A five feet long plank is divided into two parts. The length of one part is 2/3 of the other. Find the length of the smaller part.

Step 1: Identify the characters.
Length of One part - - - - - -
Length of the Other part - - - - - -

Step 2: Which part (character) is dependent on the other?
One part is dependent on the other.

Step 3: Which one is unknown.
The other part.

Step 4: Let us call the length of the other part as X feet. Remember X is the length of the other part not the name of the other part.

Step 5: Read the problem again by substituting X as the length of the other part.
"A plank five feet long is divided into two parts. The length of one part is 2/3 of X. What is the length of the smaller part?"
One part = 2/3 of X
Other part = X

Step 6: Do we have an algebraic equation involving the variable X?
No.

Step 7: Let us develop an equation. What information in the given problem has not been used?
The length of the plank.

Step 8: What is the relation between the two parts and the length of the plank?

The length of two parts together is equal to the length of the given plank.

Step 9: Let us write the equation by translating this information.

Part one + part two = 5 feet.

$$(2/3)\,X + X = 5.$$

Step 10: Now solve this equation.

What makes this equation difficult?

The fraction in the equation.

What is the common denominator of the fractions in the equation.

3.

Multiply the whole equation by 3.

The new equation looks like

$$2\,X + 3\,X = 15.$$

Now apply CLT (collect like terms.)

$$5\,X = 15.$$

Is X term alone on one side of the equation. Yes.

Is X alone in the equation.

No.

Divide both sides of the equation by 5.

You get $X = 3$.

What are the lengths of the two parts?

$(2/3)$ X and X.

Find the values of $2/3$ X and X.

$(2/3)$ 3 and 3.

Or, 2 and 3.

Now, what are the lengths of the two parts?

2 ft. and 3 ft.

What is the length of the smaller part?

2 feet.

In the introduction of the word problems involving algebraic equations, the teachers and tutors need to develop a script as detailed as outlined above and use it with students to make sure the conceptual and language development is thorough. This type of dialogue between the students and the teacher is an important part of the development of the mathematical language and the translation process. The questions that the teacher asks should be answered by all the students in the class, one-by-one.

(c) If the given problem is a ratio one, then identify the characters involved in the problem, label them and write them horizontally:

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Example: There are 5 times as many apples as oranges. Total number of fruits is 24. How many apples are there?

Step 1. Is this a ratio problem?
Yes.

Step 2. What are the characters?
Apples and oranges.

Step 3. Write them horizontally.
Apples Oranges

Step 4. Designate variables for the two characters.
Apples Oranges
# of apples A # of oranges O

Step 5. Read the word problem by replacing the characters by their variable names.
There are 5 times as many A's as O's.

A	O
5	1

$$\frac{A}{5} = \frac{O}{1}$$

Step 7. Cross multiply the above equation.

$$1\,A = 5\,O$$

Step 8. What information from the word problem, we have not used so far?
The total number of fruits is 24.

Step 9. Set up the equation using this information.

$$A + O = 24$$

Step 10. solve the equations ……

Memory retrieval difficulties play a great role in students who have difficulty in solving equations. The three things that we have found to be helpful a great deal are: Asking students to give the script of the procedure before they solve the problem; Doing at least 10 minutes of oral mathematics in the class each day; And assigning daily homework that has three major components — 1/3 of the homework should be cumulative in nature (including representative examples from everything you have done up to that point), another 1/3 exactly like the work you did that particular day, and final 1/3 work designed to involve interaction of the other two-thirds of the homework.

Remediation of algebraic equation difficulties is not an easy task. Rosnick and Clement (1981) conclude "the reversal problem is a resilient one and that students' misconceptions pertaining to equation and variable are not quickly 'taught away.' In fact, at least seven out of nine students demonstrated in one way or another that they maintained the reversal misconception, even after our attempts at remediation. Students need to develop a better understanding of the basic concepts of variables and equation. They should know how to distinguish between different ways in which letters can be used in equations. They should learn to distinguish when letters are used as labels referring to concrete entities or alternately, as variables standardizing absolutely for some number or number of things."

Rosnick & Clement (1981) experimented with several different teaching strategies with several different students to teach translation from English to algebraic equations, usually to no avail. Their conclusion was "that there doesn't seem to be a quick solution to the difficulties that students face concerning the concept of equation and variable." They tend to believe that many secondary school students and college students have not yet reached the necessary level of intellectual development to be able to make the distinction between the different ways in which letters can be used in equations. They emphasize that "we as mathematics educators, should be aware of the distinction ourselves." In teaching algebra we should make every effort to convey that distinction. It is important that we stay aware of the difficulties that our students are having in trying to understand labels, variables, constants, parameters, and all other uses of the variables in algebra. It is also important that we become aware of all the conceptual pitfalls to which our students can succumb.

Language of mathematics, particularly translating from English to algebra and back is a separate (separate from the mathematical part) and is an important skill. To acquire this skill extra instructional and remedial time is needed not only to understand it but mainly to apply it. Specific exercises must be planned so students do not practice just syntactic translations. It is important to point out and show when syntactic translations do and don't work.

Remedial activities have included several approaches some of them are:

1. Simply telling the students that the reversal is incorrect.
2. Telling the student that the variable should be thought of as "number of students," not "students."
3. Pointing out (with pictures) that since "students" is a bigger group than "professors," one must multiply the professors by six to create an equality.
4. Asking the students to draw graphs and/or tables.
5. Specifically showing the students how to set up a proportion to solve the problem.
6. Asking the students to test the equations by "plugging in" numbers.
7. After the student has generated the algebraic equation to the given word problem ask the student to read it back in English. If the two statements coincide he has done it right. If there is any discrepancy ask the student to analyze the reason for this discrepancy.

8. Demonstrating a correct solution to the students, using an analogous problem.

VI. Sex Differences in Students' Abilities to Solve Word Problems in Algebra

In a recent study (Phillips, Uprichard, & Blair, 1983) designed to determine if sex differences exist in mathematics performance which required high level cognitive tasks such as ability to solve word problems in algebra in subjects who had completed the same number of mathematics courses and had equivalent mathematics achievement and reading ability, found that:

1. Males are significantly better than females at solving word problems in algebra.
2. General mathematics ability is more of a contributing factor for girls in problem solving performance in algebra than it is for males.
3. Reading ability is more of a contributing factor in both male and female ability to solve word problems in algebra than general mathematical ability.
4. Although males have a more positive attitude toward mathematics than females, attitude is not a major contributing factor in the ability to solve word problems in algebra.
5. Perceived ability in mathematics is a crucial variable in word-problem-solving-ability especially for females.

"The investigators feel that this may be the most important finding of [this] study... [If one believes] that one has the ability to do a task leads to an increased likelihood of performance, it would seem that the solution to female participation and achievement in mathematics at [higher level] is a simple one — change female's perceptions about their ability." This statement is not only true for females but it applies to all students. "But changing one's perceptions about his/her ability may be an internal and therefore relates more directly to cognitive rather than affective variables (Schunk, 1981)." Thus, we agree with the findings of this team of researchers "... that programs designed around successful mathematics experiences would have better payoff than programs designed around other variables." By practicing the development of the language of mathematics, translation ability from one language to another, applications of mathematics to real world situations, and mathematical skills one would be able to gain success in solving word problems in algebra.

VII. Predicting Eighth-Grade Algebra Achievement

Another aim of improving instruction in algebra is to improve the performance of students who take algebra in the eighth grade. As we mentioned earlier some of the students, who were doing quite well up to this point, begin to do poorly in the eighth grade algebra. Their difficulty is a special one. In most cases they are very intelligent, but they may not have the abstract reasoning and cognitive abilities needed to prepare them for the algebra course at this age. A number of researchers have investigated the relative value of previous grades, ability measures, achievement tests, and algebra aptitude tests for predicting success in a beginning algebra course. On the one hand, the research reported in the literature has shown previous mathematics grades to have the highest correlation with algebra grades, whether the set of predictors included only conventional ability and achievement tests (Rothenberger, 1967) or a special purpose aptitude test as well (Barnes & Asher, 1962; Kovaly, 1980; Mogull & Rosengarten, 1972, 1974). On the other hand, two studies that included algebra achievement test scores as a second criterion of success found algebra aptitude tests to be the best predictors of both criteria (Hanna, Bliegh, Lenke, & Orleans, 1969; Sabers & Feldt, 1968). It is important to note, however, that previous grades are not among the independent variables in the later study.

In the light of the studies cited above, neither performance on mathematics achievement tests nor general intelligence surfaced as the single most important predictor of algebra achievement. Total mathematics achievement test scores (Barnes & Asher, 1962; Sabers & Feldt, 1968) and computation subtest scores (Rothenberger, 1967) were shown to correlate substantially with algebra achievement. And Taylor, Brown, and Michael (1976) found that cognitive ability measures are better predictors of attainment in an algebra course than a host of affective and demographic variables.

Because algebra prediction studies have focused primarily on courses taught in the ninth grade, very little is known about the factors that may be related to achievement among the academically talented students who are enrolled in algebra courses in the eighth grade. Flexer (1984) in her study of the performance of academically talented eighth graders enrolled in an algebra course found that an algebra prognosis test was identified as the best overall predictor of success in an eighth grade algebra course. Scores on the prognosis test provided the best indication of algebra grades and were just slightly less effective than problem solving scores in predicting achievement on a standardized algebra test. The study gave further evidence that the prognosis test could be used as a discriminator between those students who successfully completed the course and those who dropped out or failed. Thus, it might be inferred that despite the additional time and expense involved with administering such a test its use for the purpose of selecting students for an eighth grade algebra course is warranted. Flexer argues that it provides a more objective criterion than previous grades alone when explaining selection decisions to parents. Of course, she warns that the setting of an arbitrary cutoff point poses a dilemma, however, since a criterion stringent enough to prevent most of the potential failures is likely to exclude a number of students who would succeed. In her studies she found that the prognosis test score and the mathematics grade for the previous year provided optimal prediction of the algebra grade, whereas the problem solving test score, prognosis test score, and IQ, in combination, best predicted performance on the algebra achievement test. This difference is explained by the fact that teacher-written algebra tests and standardized algebra achievement tests assess somewhat different domains of competence and knowledge. The findings further suggest that some of the difference may be in the standardized tests' stronger emphasis on problem solving. This suggests, perhaps algebra achievement is related to the ability to reason at the formal operational level described by Piaget (Inhelder & Piaget, 1958). Research has indicated that even in samples of above-average intelligence no more than half of the eighth graders can be classified as formal operational (Flexer & Roberge, 1983; Roberge & Flexer, 1979). Research has shown that in high-ability eighth graders there is a relationship between level of operativity on a Piagatian test and mathematics problem-solving scores (Roberge & Flexer, 1983). It seems, therefore, to use some kind of reasoning or problem solving test (may be a paper-pencil Piagetian test) to be included in order to select and predict the performance of eighth graders in algebra course and algebra achievement tests.

VIII. IMPORTANT MATHEMATICS SKILLS AS A PREPARATION FOR ALGEBRA

Although we have already identified, earlier, the mathematics skills that a student enrolled in algebra needs. We find that along with the general basic algebraic skills, a few key mathematics skills are used more often than others. In this section we shall discuss these key concepts.

The Number Line and Its Application

One of the major difficulties students have in algebra is manipulating the signed numbers (integers) in the context of operations with integers and algebraic equations. The difficulty arises in the interpretation of signs. Typical disabilities that affect progress in algebra are difficulty with visual perception, spatial organization, closure (Bley & Thorton, 1983), and lack of understanding of the concept due to the lack of a consistent model. For many students, the greatest difficulty in dealing with integers is misreading or misplacing the sign. For this reason, it is essential that the teacher be consistent in the placement of signs used with integers. When signs are to be interpreted as negative or positive, they should be placed in a **raised** position. When used to denote an operation, the sign should be centered between the two numbers that are involved. Unless understood, 7 - 8 creates a difficulty. A typical question from the students in that situation is: Does the sign in front of the 8 mean "subtract" or "negative?" Am I supposed to subtract here 8 from 7 or add - 8 to 7? When the signs appear in conjunction with variables in the problem for many students the problem becomes acute. For example:

$$x^2 - 5x + 6 = 0;$$

$$x = - (- 4) + - ((- 4) 2 - (4) (- 3) (- 5)) (1/2/) (4) (- 3); \text{ etc.}$$

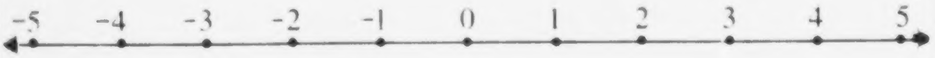
Dealing with so many signs is a difficult task for many students.

○ **Signed Numbers: How to Manipulate Plus and Minus Signs**

The plus and minus signs in arithmetic are generally taught as operations of addition and subtraction respectively, but in algebra "-" is regarded as the reverse of " + ." Any number with a " + " before it is a gain, increase, asset, or lies on the right of zero on the number line, while any number with a "-" sign before it is a loss, decrease, debt, or lies on the left of zero on the number line. If + indicates a measurement along the scale in one direction, then "-" indicates measurement along the scale in the opposite direction. If + 17 means a temperature of 17 above zero, then - 17 means a temperature of 17 below zero. If +500 means a deposit to an account (credit), then - 50 means a withdrawal from the account (debit). If 1,500 means 1,500 ft. above sea level, then - 1,500 means 1,500 ft. below sea level. If +568 means A.D. 568, then - 568 means 568 B.C. If +10 means 10 paces forward, then - 10 means 10 paces backward. Thus the numbers in arithmetic are representations of quantity, whereas, the number with a sign attached are **directed** or **signed** numbers.

The procedure for teaching signed numbers is a complicated one in most textbooks; most textbooks devote several sections over hundreds of pages and teachers devote several weeks, some times months, on the teaching of integers in the seventh or eighth grade and again in 9th grade or during the first year of college algebra. Still many students come to the algebra course ill prepared in this topic. In most cases the teacher has to review this material again and again. There is always a tug-of-war between the time element and the students' need to master the operations with integers. The following procedure is given here to deal with this topic in such a way that the least amount of time is spent on this and we get maximum level of proficiency on the part of students.

To help understand numbers and their uses, many models have been used. One of the most successful of these ways to picture numbers is the use of the number line. The number line is used in many forms in our everyday life. A ruler is a fine example, of course. The common application of a number line appears in a thermometer, comparison of profits and loss in a financial statement, and many other measurement devices. Graphing in two-or three-dimensional coordinate systems is an extension of number line application, where two or three number lines are used as axes.



Applications of Number Line

Step One:

Observe the following - −8 or - (- 8). There are two signs before the number 8. Similarly in the following examples also there are double signs.

− +6.4, + −56, − +3xy, + +72abc, − −8 a/bc, etc.

For the moment suspend your definition of signed numbers that is first sign before the number indicates the operation and the second sign is the sign of the number. Let us redefine them slightly differently.

"+" sign in the first position will be treated to give the command to the imaginary person standing on the position "0" on the number line (you can actually ask a student to stand at zero on a number line drawn on the floor of the classroom to make this demonstration practical) to 'go in the right direction on the number line';

"-" sign in the first place will indicate the command to the imaginary person standing on the position "0" to 'go in the left direction on the number line';

"+" sign in the second place will give the command 'continue doing what your first command told you to do';

"-" sign in the second place will give the command 'do just the opposite of what your first command told you to do.'

Let us now apply these definitions to the following situations:

+ + means that the first + tells you to go in the right direction of the number line and then the second + tells you to continue in that direction which means that you are going in the right direction of the number line showing the net outcome as the right direction which means that + + results in +. Therefore, we have

+ + = +.

- - means that the sign '-' in the first place tells you to go in the left direction of the number line and then the second '-' tells you to do just the opposite, in other words reverse the direction, so you are now going in the positive direction (towards the right) of the number line showing the net result as the right direction which means that - - results in +. Therefore, we have

- - = +.

+ - means that the '+' in the first place tells you to go in the right direction of the number line and since '-' is in the second place it tells you to reverse the direction of the first command it makes you to go in the opposite of the first direction, that is, you are now going in the negative direction of the number line the net result being that you are now going in the left of zero on the number line which means that + - results in -. Therefore, we have

+ - = -.

Finally, - + means that the '-' (being in the first place) tells you to go in the left direction of the number line and since the sign '+' in the second place, it tells you to continue whatever you are doing (that is to continue in the left direction) as the result of the first command. So the final direction you are now going is in the left direction of the number line. That results in - + being equal to one command -. Therefore, we have

- + = -.

In summary, as a result of this we have

Rule 1 + + = +, - - = +, + - = -, **and** - + = -

Rule 1, is the most important rule which means that whenever you see a double sign you change it into a single sign using the above rule.

For example:

- - 4 = + 4, + - 7 = - 7, + + 7.5 = + 7.5, etc.

+ - 7 + + 4 = - 7 + 4, - - 4 + + 6 = + 4 + 6, etc.

○ **Operations of Signed Numbers**

A. Consider the following examples:

(a) + 6 + 2 = ?
(b) + 6 - 2 = ?
(c) - 6 + 2 = ?
(d) - 6 - 2 = ?
(e) + 6 . + 2 = ?
(f) + 6 . - 2 = ?
(g) - 6 . + 2 = ?
(h) - 6 . - 2 = ?
(i) + 6 ÷ + 2 = ?
(j) + 6 ÷ - 2 = ?
(k) - 6 ÷ + 2 = ?
(l) - 6 ÷ - 2 = ?

In examples (a) and (d) the operation or the sign before both the numbers is the same indicating the same activity. For example, if the temperature was + 6 degrees then it went up + 2 degrees what will be the net result? Naturally the answer is + 8 which we obtained by adding. Similarly, if the temperature was - 6 degrees and it went down 2 degrees what will be the result? We find the new temperature is - 8 degrees. Once again the final answer was found by adding the two numbers. Or moving in the same direction in both instances, therefore in such situations we apply the following rule:

Rule 2. For the same signs we add the numbers and use the common sign.

For example: + 6 + 2 = + 8, - 6 - 2 = - 8

B: In examples (b) and (c) the operations or the signs before the two numbers are opposite indicating opposite activities. In such situation we apply the following rule:

Rule 3. In the case of opposite signs we subtract the numbers and use the sign of the larger number.

Thus + 6 - 2 = + 4, - 6 + 2 = - 4

In multiplication and division we have the following rules.

Rule 4.	+ . + = +	- . - = +
	+ . - = -	- . + = -
Rule 5.	+ ÷ + = +	- ÷ - = +
	+ ÷ - = -	- ÷ + = -
Thus	+ 6 . + 2 = + 12,	+ 6 ÷ + 2 = + 3,
	+ 6 . - 2 = - 12,	+ 6 ÷ - 2 = - 3,
	- 6 . + 2 = - 12,	- 6 ÷ + 2 = - 3,
	- 6 . - 2 = + 12,	- 6 ÷ - 2 = + 3.

The above rules in the form of a chart should be provided for reference. Ask the students to memorize the chart. The following procedure is found to be very helpful.

Step 1: Write about 25 problems dealing with operations of integers. First, ask your students to convert double signs into single signs only in all the 25 problems. You should ask students one by one to do this. This should be done orally. You write the answers to each problem written on the chalkboard. You should also ask them to give the reason for the answer. When the student answers a particular question you should ask him to quote the rule being used.

Step 2: Now we have all the 25 problems written involving single signs. Now ask which rule (rule 2 or 3) will apply. Just ask for the sign of the final outcome. You write the sign of the answer using rules 2 and 3 for each problem on the chalkboard. Ask the student to state the rule when he/she gives the answer. Repeating the rule in this manner will help them to memorize the rules. This is one of the reasons that all this work should be done orally. The objective is to automatize the process of operating with the integers.

Step 3: Now we have those 25 problems on the chalkboard with the sign of the answer in each case already identified. Now ask students to identify which rule to apply (to add or subtract). And ask them to quote the rule orally.

Step 4: Solve several problems from start to finish. Ask students to quote the rules as you proceed in the solution of the problems.

Other concepts that are useful in reviewing before we begin algebraic work are ratio and proportion, order of operations, properties of numbers.

Conclusion

Mathematics is one of the principal disciplines to have determined the evolution and accelerated development of science and technology. For this reason science places primary emphasis on the development of mathematical theory and on improving its effectiveness when applied to practical ends. Dealing with this problem requires that due emphasis in mathematical studies be placed on the exposition of mathematical ideas, concepts and methods together with their applications to real-life situations. Algebra word problems provide a very good example for such an application.

Development of scientific thinking in students is facilitated by such instruction techniques as work with models, creation of problem situations, as well as an approach whereby the proof of a number of facts is preceded by observations involving the identification of general patterns and the statement of hypotheses (these hypotheses should be formed by the students themselves).

The use of models in algebra courses are one of the most effective devices to understand the concepts and their extensions to other higher ideas. Models can be used to explain concepts or to demonstrate proofs of important results and to solve problems. In the shaping of concepts the successful use of models enables students to find the required definitions independently. All that is left for the teacher is to direct the students to arrive at the definitions in logically correct forms.

There should be a general reorientation of mathematical education aiming at mathematics which is a useful tool for the majority of students. The teaching of mathematics as a means of solving multidisciplinary problems by using modelling methods should restore student interest, show mathematics as being useful, enrich students' knowledge of related subjects and to enable them better to memorize mathematical formulas and methods, encourage logical reasoning and allow more students access to a higher level of mathematics. As education has become increasingly universal, however, students of lesser ability, and with more modest vocational aspirations and daily life requirements, have entered the school system in greater numbers. A major problem results when these students are exposed to problems that do not have any resemblance with real life and meaningful applications.

A Japanese study in 1955 was concerned with people who had learned their mathematics before the period in Japan in which mathematics teaching was focused on daily life experiences and before compulsory education was extended to secondary schools. Although it was found that most people retained the mathematics skills and knowledge well, rather fewer claimed that this material was useful in their work. This indicates that we need to make the applications relevant and accessible to a larger group of students.

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REVERSAL PROBLEMS IN MATHEMATICS AND THEIR REMEDIATION

Mahesh C. Sharma

The accurate perception of letters, numbers, words, phrases, geometrical shapes, and diagrams (any visual information) is the first step in the conceptualization process of ideas in a culture with a heavy emphasis on the visual presentation of information. To his perception the child must also add the understanding and interpretation of these symbols. But this understanding is difficult and even impossible to achieve if the child's perception is inaccurate in the first place. The inaccurate perception and faulty information processing results in the misunderstanding of the concepts.

Reversal of numerals, letters, and geometrical shapes is a common occurance in children's work in the early grades. Generally, with time and with maturity (physical, neurological, psychomotor), this problem disappears in most children by the second grade and generally by third grade. These reversals tend to gradually disappear as the stability of the perceptual process increases with maturation. In many severely learning disabled individuals this condition persists, apparently as a part of a manifestation of general disorganization. In most young children the reversal problem disappears with development and maturity. And in a small number only they persist. In some cases even into adulthood. The persistence toward reversals in the teenage and adult population may reflect an ongoing perceptual instability. This needs to be addressed through a well planned and well executed perception remediation program. Such a program should include activities so that the negative effects of perceptual deficit on language development and mathematics conceptualization may be minimized.

Lynn (1963) cites an experiment by Nelson (1955) which demonstrated that four to four-and-a-half year olds could discriminate mirror images (b and d, n and u, etc.) better after thirty minutes of training than five year olds who received no special training. Nelson concluded from this that 'the existence of this disability at five must be taken as being due entirely to lack of experience in the practice of this concept.' However, there are some children who continue to reverse their letters, numbers, and geometrical shapes in the third grade and in some cases one can spot it even in the later grades. There are adults who exhibit perceptual difficulties in mathematical activities where perception and spatial orientation and space organization are key elements and that means most of mathematics.

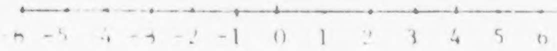
Reversals, inversions, transpositions and substitutions of numerals, along with similar behaviors with alphabets, persist in the dyslexic child long after such behavior is developmentally normal (Ilg and Ames, 1965). But the key problem that we plan to address in this issue is the difficulty children face in number conceptualization because of their difficulty with perception, particularly visual perception.

Reversals in numbers and letters sometimes are also dependent on a child's lack of skill in identification and orientation of left and right. Therefore, the development of left-right orientation and discrimination has an important role in the remediation of reversal problems. Binet's **(Standford-Binet Test of Intelligence)** pinpoints six years of age as the time at which a child can indicate his left hand and his right ear. In schools in Argentina, for example, the majority of children are able to recognize their right and left around seven years of age (deQuiro's & Schrager, 1978). According to Jean Piaget, the acquisition of the notion of left and right, in the sense of relative notions, goes on through three stages corresponding to three "desubjectivations" (that is, to three progressive socializations of thought): the **first stage** (five to eight years), in the course of which left and right are only considered from a child's point of view using his/her body as a reference; the **second stage** (eight to eleven years), when they are also considered from the point of view of others and of the questioner (mirror image persepective); and the **third stage** (eleven to twelve years), which marks the time when left and right are considered increasingly from the point of things themselves.

In the first stage the child uses his body as a reference to identify the relative spatial orientation of objects in the environment. The second stage is called the mirror image or translation of orientation stage. In this stage, the child is able to translate the teacher's demonstration and material from the chalkboard to his own setting. This is crucial from the mathematics learning point of view. For example, when the teacher demonstrates that the numbers on the right (she points to her right) are positive and increasing. Or when she demonstrates that the place value increases in this direction (points to her left), these teacher demonstrations need to be translated by the child to his own context. He needs to see it translated to his own reference of left and right orientation. If he is not able to do so, then he has difficulty with related concepts.

It is quite a common observation on the part of second and third grade teachers that many children have difficulty with the place value concept. Many teachers have reported that they find the understanding of the concept one day and a few days later the child has problems with the concept. One reason for this difficulty is that the teacher works with the child on place value but does not check whether the child has acquired left-right orientation or not. If the child has not acquired left-right orientation, the instructional and remedial work with the place value will be short-lived. The understanding of the place value concept will be dependent on his mastery of left-right orientation. Generally, the error relates to reversal of place values. Remediation, therefore includes not only work with place value concept but also on the development of left-right orientation. In the third stage, there is no fixed reference point. The reference orientation is dependent on the situation and context. For example, when the teacher says, "the hypotenuse in a right triangle is opposite to the right angle," the reference for orientation in this case is the right angle. Or when she says, "Sine of an angle in a right triangle is equal to the ratio of the opposite side and the hypotenuse." Once again the reference orientation is the angle under discussion. Generally, when the right angle or the diagram in question is in its standard position, it is easier for the student to perceive and understand the concept, but when the right triangle is not in the standard position the student has difficulty. In this situation, the student has to focus on both the absolute orientation and the relative orientation. The students who identify the hypotenuse incorrectly (pointing to the slanted side always) are paying attention to the absolute position rather than the relative position and orientation. At this point, it is important to help the student analyze the definition and make him aware of the relative and absolute positions of components in the definition and the diagram.

Figure 1a



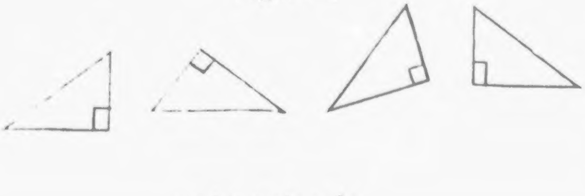
Number line

Figure 1b

thousands	hundreds	tens	ones

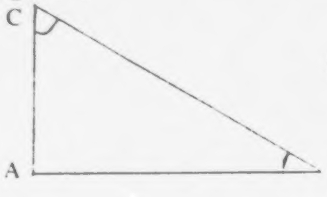
Place value columns

Figure 2



right triangle

Figure 3



$$\text{Sin } B = \frac{AC}{BC}$$

$$\text{Sin } C = \frac{AB}{BC}$$

But the student writes

$$\text{Sin } A = \frac{AC}{BC}$$

trigonometric relationships

Kaliski (1967) has noted that many of the symptoms that identify children with learning disabilities can be related to arithmetic disabilities. Disturbances in spatial relationships, visual perception abnormalities, and difficulty with symbols are among the reasons for mathematics learning problems in general and number difficulty in particular.

Before planning and implementing activities for children who consistently reverse their letters and numerals one should try to find the reason and cause of this problem. Apart from the lack of maturity, poor fine motor skills (as seen in handwriting and drawing) and general intellectual skills, the child who reverses letters and numerals may do so for one or more of several reasons that include among others:

- Difficulty with visual perception
- Difficulty with spatial orientation space organization
- Short term visual and auditory memory
- Confusion with similar looking patterns due to both visual perceptions and spatial relationship
- Introduction of writing of numbers and letters to children before enough visual stimuli (with diversity in quantity and quality) has been received
- Disturbance in body image

A. DIFFICULTY WITH VISUAL PERCEPTION

The word perception refers to the recognition of sensory information produced by different stimuli coming from the outside world. It is the resultant sensory-motor patterns produced by sensory stimulation. Perception, therefore, can be considered an acquired skill: the ability to interpret or give meaning to what is seen, heard, or touched.

Several constructs of perception have implications for children with learning problems: the perceptual-modality concept, overloading of the perceptual systems, whole and part perception, visual perception, auditory perception, haptic perception, cross-modal perception, form and directional perception, and social perception. Visual perception is literally the capacity to interpret, or give meaning to what is seen. Since perception is an acquired skill, the teaching process can have a direct impact on the child's perceptual facility. Selection of appropriate teaching procedures can improve various subskills of perception or modify the learning process in the light of the child's perceptual abilities and disabilities. Within the broad scope of visual perception can be identified. Chalfant and Scheffiin (1969) identify the components as: spatial relations, visual discrimination, figure-ground discrimination, visual closure, object recognition. **The Frostig Developmental Test of Visual Perception** evaluates these five subskills that sample the child's ability in visual perception. **Illinois Test of Psycholinguistic Abilities (ITPA)** has several subtests that assess aspects of visual perception: visual perception, visual association, visual closure, and visual sequential memory. Subtests of **Detroit Tests of Learning Aptitude** sample aspects of visual perception including pictorial opposites, visual attention span for objects, memory for designs, and visual attention for letters.

Other tests of visual perception are: the **Bender Visual-Motor Gestalt Test; the Development Test of Visual-Motor Integration; the Motor Free Test of Visual Perception and Southern California Figure Ground Perception Test.** Subtests of the **Weschler Intelligence Scale for Children (WISC)**, provide valuable information about the child's skills in visual perception tasks.

B. VISUAL CLUSTERING

Many young children who have a basic arithmetic disability are unable to see objects in groups or sets — an ability that enables others to quickly identify the number of objects in a group. This ability of identifying the cardinality of a cluster of objects (group/set) visually, called **visual clustering**, is the key in number conceptualization and therefore in all arithmetic operations. Visual clustering is an ability to identify the cardinality of a small collection of objects by visually approximating it, that is having a feeling for the numerosity of the group without counting each object in the cluster one by one. To better understand number concepts and as a preparation for arithmetic operations a child must be able to perceive the number of objects in a group (as long as the number of objects is small and there is regularity in the arrangement of the objects in the collection) as a result of their configuration without counting them. In this process a kind of mental regrouping of objects in clusters does take place even when the arrangement is not displayed in the regular clusters.

This mental regrouping and mental manipulation of objects is the outcome of mentally forming patterns of frequently presented regular patterns in real life experiences such as the child has when he plays with die, dominos and playing cards. By frequently seeing the dice, the child can form stable visual images of these visual patterns or clusters in the standard or regular format. Later on he will be able to recognize the cardinality of a set where the objects may not be arranged in the standard patterns by just visually manipulating and rearranging the objects.
(Standard patterns here relate to the organization of objects in the pattern of dots seen on a die and the symbols on playing cards.)(see figure 4)

Visual clustering is the integration of the visual image of the cluster (physical representation of the objects), the shape of the number representing the collection, and the cardinality. In other words, when he sees the number the visual image of the appropriate cluster in a regular pattern comes to his mind and when he sees the visual cluster the shape of the numeral flashes in his mind. The following real life situation can explain the importance of this ability. Anyone who has played cards realizes that one can identify a card in a fraction of a second without counting the objects on the card. Sometimes this is done even when only a portion of the card is visible. This takes place when we have formed stable images of these regular patterns.

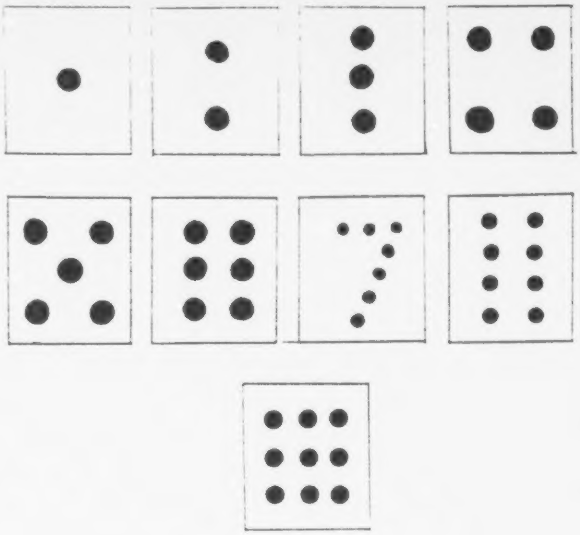


FIGURE 4

This identification of number is generally done by recognizing the visual clusters. Very young children can recognize ordinary playing cards only by counting the arrangement of objects on them. The major skill utilized in this is sequential counting. When a child can recognize the number associated with pictorial information on the playing card by just looking at it then he has processed the information visually in a correct manner. The child with very good ability in identifying the visual cluster finds it easier to perform the arithmetic operations. On the other hand the child unable to identify visual clusters has real difficulty with arithmetic operations. For example, even when adding a group or set of three objects with another set of four objects, children having difficulty with arithmetic begin by counting the objects one by one to determine the total number in the sets rather than visual clustering.

The prevalent method used by most mathematics textbook series in elementary schools is to present objects in random grouping and linear arrangements. In a recent study by Dunlap and Hynde (1981) relating to grouping in mathematics textbooks they found that in few "instances elemental groupings were used to represent a particlar number; however, the use appeared to be incidental rather than intentional. None of the examined textbook series used elemental groupings of objects into groups of one, two, three or four objects as the prevalent method of representing the numerals 1-9. (Here, the elemental grouping refers to regular patterns). A common characteristic of the textbooks was the use of one grouping pattern to introduce a numeral; and, then, a variety of patterns were in the exercises that followed. The illustrations did not provide a basis for identifying a set for a specific number from memory. The child must resort to counting the objects without a real understanding of number and without developing a mental image of the collection to associate with the number. The sequential counting process beyond age 7 actually interferes with the child's development of a mental image to associate each number." Actually an undue emphasis on sequential counting and lack of experiences with visual clustering results in the continued use of fingers for counting and numerical operations.

In the initial stages of creating the mental image of a number a child needs to integrate the following:

(a) visual cluster (grouping pattern)
(b) shape of number
(c) cardinality (count) of the grouping pattern

When the shape of the number and the visual cluster representing the numeral are superimposed as in the figure 6, child achieves this integration. For example, children who have difficulty with visual perception when introduced to number concepts by using linear and non-uniform visual representations of the numbers are the ones who keep using the counting procedures (such as finger counting) long after these procedures are acceptable and efficient. The remedial procedural steps are detailed in the remedial section of this article.

C. SPATIAL ORIENTATION AND SPACE ORGANIZATION

The initial concepts of spatial relationships are normally acquired at the preschool level. Kaliski (1969) reports that the child with an arithmetic disability often confuses spatial relationships such as: up-down, over-under, top-bottom, high-low, near-far, front-back, beginning-end, and across. Strauss-Lehtinen (1947) found that a disturbance in a spatial perception interferes with the visualization of the entire number system. The child may reverse numbers, or may not perceive distances between numbers accurately. Children with poor quanititative abilities frequently perform inadequately in visual-motor tasks. Because they cannot capture the gestalt of a shape and because they have a disturbance in perceiving spatial relationships and in making spatial judgements, they may not be able to copy geometric forms, shapes, numbers, or letters adequately.

D. SHORT-TERM AUDITORY AND VISUAL MEMORY

Sensation and perception take place when the stimulus is present; and as long as the stimulus is present it's an ongoing activity. Memory pertains to sensations and information already received and perceived. The ability to retrieve previously experienced sensations and perceptions when the stimulus that originally evoked them is no longer present is called memory. There are at least three stages of memory in which the child with learning problems in mathematics could have difficulty: reception, storage, or retrieval.

Researchers in the field of human memory refer to the three types of memory storage systems as: (1) a sensory storage of register, (2) a short-term store, and (3) a long-term store. Information is first contained in the storage register; then it is identified and retained in the short-term store, finally selected information is assimliated into what already is stored in the long-term store. Information that is not coded successfully for long-term storage fades away and is largely forgotten. To be able to commit a piece of information to the storage register, one has to insure that the new information bears a good relationship with some part of old knowledge, skills and experiences. On the other hand, rehearsal is needed to maintain the information in short-term store and to transfer it from short-term to long-term store.

Many children with learning disabilities have difficulty recalling things that looked alike or sounded alike. The inability to remember word, numbers, formulas, directions, and explanations are examples of disabilities of remembering information. Memory problems can also be related to specific perceptual modality, e.g. visual-memory or auditory memory. For efficient learning, and individual's memory in many areas of factual performance must become an automatic, habitual response to a stimulus. Many factors have an effect on memory: the child's intensity of attention, the child's cognitive preparation for the concept to be learnt, meaningfulness of the material, interest in the subject, relationship of the new concept with old experiences, and the amount of drill and overlearning. The importance of review, organization of the material and memory strategies cannot be overemphasized.

The child's reversal problems whether in arithmetic or in reading and writing may not necessarily arise because of all the above mentioned factors and prerequisite skills. Moreover, these factors and skills are not mutually exclusive. One should remember that the problem exists only if the child is reversing the letters and numerals consistently. The infrequent presence of reversals in a child's work does not indicate a serious problem. Actually, in some cases, according to Downing and Thackray (1971), "the tendency to confuse d and b, n and u, etc. is more likely due to a good perceptual development rather than the opposite because the child's experiences prior to coming to school will have taught him to ignore such minor images." Children's experiences before school are much more general: as they begin to engage in more formal learning they begin to discern subclassifications in bigger classes. For example, a child knows that a walking stick is still 'a stick' no matter in which direction the handle is pointing, but when he learns to read he must learn that in reading and writing direction does matter. The letters b and d are not both sticks with a handle. The one pointing to the left represents one sound, the one pointing to the right means something different and while the one with a slightly modified handle on the top (9) or bottom (6) means somethings entirely different. The slow reader who continues to make reversals has failed, it seems, to unlearn the generalizatons he had learned before coming to school; now he has to qualify his generalization and create and distinguish subclasses. A child's focus on whole rather than the parts of the whole is behind this difficulty. For proper conceptualization, the child needs appropriate focus on whole-to-parts and also on parts-to-whole. The focus on parts-to-whole by children can be developed by asking pointed and probing questions relating to the tasks at hand.

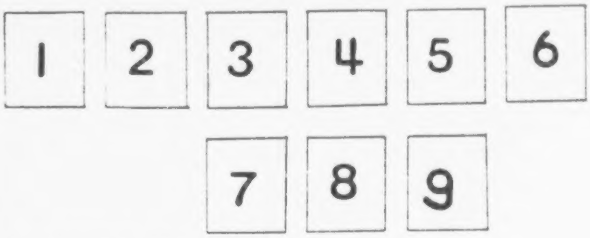
E. SUGGESTIONS FOR REMEDIAL ACTIVITIES

1. A first step in the development of number concepts in their proper form and aspects is the ability to focus upon and recognize a particular type of object. Have the child compare and contrast objects of different sizes. The teacher should ask the child to repeat visual patterns using attribute blocks. (Attribute blocks are geometric shapes of varying sizes, colors and thickness.) One should begin with two or three pieces of different colors and prepare a pattern and ask

the child to select the pieces and copy the patterns. Puzzles, pegboards, geoboards, or form boards can be used to make different shapes and geometrical designs to help the child focus on shapes and spatial relations. The child having difficulty in identifying these shapes and spatial relations must be given verbal clues about the shape and once identified the child should describe the piece by feeling the edges and looking at the object. Once the child has copied pattern ask him to describe the pattern assuming different perspectives: from his position, from your position, and then from any other position. After this, change the pattern by rotating it and ask the child to follow your actions. Observe the direction and the way the child rotates the pattern. Ask the child to bring the pattern back to the original position. (This ability is not only good for spatial orientation/space organization and visual perception but also for developing reversibility of thought which is an essential skill for mathematical conceptualization.)

2. A modified deck of playing cards (see figure 6) can be used to help children develop **visual clustering** and provide opportunities of integrating the collection with the represented image of the number. Recognition of numbers and writing of numbers without reversals is helped if a child can integrate the physical act of counting (tactile experiences), representation of counted results (writing of numbers - visual experience) and identification recognition of symbols (recall or auditory experience).

FIGURE 5



REMEDIAL PROCEDURES

Step 1: Show a collection of concrete objects arranged in elementary patterns (regular visual clusters as represented in figure 4.) Or show a card which has collection of dots arranged in regular visual clusters. (See Figure 6)

Step 2: Show the collection of cards with the shape of numerals drawn on them. (See Figure 5)

Step 3: Ask the child to select a card from the collection shown in Figure 4 and ask him to superimpose the shape of the numeral on the cluster of objects matching with the number. (The two sets of cards shown in Figures 4 and 5 should be prepared in such a way that they can be superimposed.)

Step 4: Now do just the opposite, show a card with the shape of a numeral drawn on it and ask the child to select the card with the appropriate visual cluster on it and ask him to superimpose the corresponding visual cluster on the shape of the given numeral.

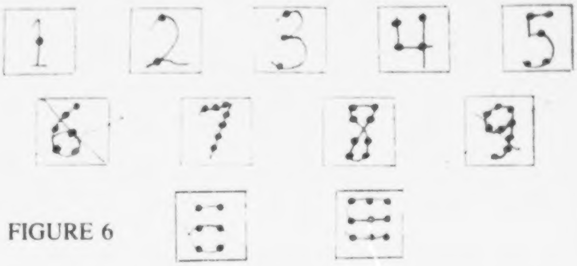


FIGURE 6

These steps should be introduced using a wide variety of concrete experiences, associating the number words and shapes with visual clusters. For example, prepare one set of 9 (3" x 3") cards of transparent material by placing colored dots (1/8" - 1/4" diameter) arranged in regular patterns as shown in figure 4. Another set of 9 (3" x 3") cards of transparent material by writing numerals in a light but clearly visible color. The numbers and the dots should be placed in such a way that dots fall on the number. Another 2 sets of nine cards prepared as above except that the dots this time are lighter and the numbers darker or vice versa.

The first set of cards are to be used for step 3 and the second set of cards for step 4. The use of many varied concrete experiences of manipulating and arranging them in standard visual clusters is needed. Children should be given plenty of time to master each number. It should also be remembered that children need to view complete sets of objects to represent a number as they associate a number word or visual symbol with that cluster.

3. Finding missing parts or given shapes in a given picture is an excellent exercise for visual figure ground and visual closure.
4. Make a deck of cards (about 50 in number with more than one card for each pattern) with patterns and diagrams as in figure 7.

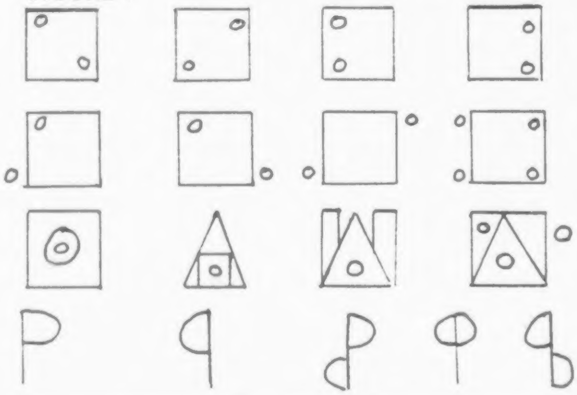
A teacher could engage the child in the following three major activities using the cards described in figure 7.

Step 1: Pick a card from the collection and ask the child to pick out the card from the collection that matches the card being shown.

Step 2: Draw one of the pictures from the deck on a sheet of paper and ask the child to observe you while copying and ask him to trace the diagram with his finger in the same direction as you drew it.

Step 3: Ask the child to copy a diagram from the set of cards on a sheet of paper.

FIGURE 7



Step 4: Repeat Step 3, but this time the teacher draws a card from the collection and shows it to the child for ten seconds and then removes it from his view. The child is then asked to copy it from memory.

Step 5: Write letters and numerals and ask the child to copy them on a sheet of paper.

Step 6: Repeat Step 5, but this time the teacher writes it and shows it to the child for ten seconds and then removes it from child's visual field and the child writes it from memory.

Step 7: If the child makes a mistake, show the child the original one and ask him/her to compare his/her drawing with the original and to find his/her mistake by asking question such as:

> Does your picture look like mine?
> If not, what is the difference?
> Can you correct it by making another one just like mine, etc?

Step 8: Practice. Practice. Practice. What then children need is practice. Ask him to point out similarities and differences between different pictures. Automatization is possible only by careful and regular reinforcements. Through practice, the learning is consolidated (Rock, 1958). For many children repetition is the key. But the "trick" is to present this repetition in many ways-kinesthetically, visually, aurally and eventually abstractly.

The other activities that help children with reversal problems are:

F. MOVEMENT AND PHYSICAL EXERCISES

Even simple games that children play are helpful in the remediation of the reversal problem. Some of the most significant examples of the importance of motor activity for the proper achievement of perception, which plays such an important part in learning, can be found in papers by R. Held as cited in Abercrombie's (1968) work. According to Abercrombie, the fundamental conclusion of Held's work relating to movement and learning are: (1) that active body movements are important for perceptual skills and (2) that the ability to compensate distorted visual or auditory perceptions depends on movement. Held's research also supports the idea that visual information and auditory information both require normal movements in order to be normally achieved. Playing games but particularly those games which make children realize the configurations of letters, numbers, geometric shapes are, therefore, extremely useful.

G. TRACING LETTERS, NUMERALS, AND GEOMETRICAL DRAWINGS

The brain receives lots of information through 'preprioceptors' (the term 'prepriorcepters' was created by C.S. Sherrington: and refers to receptors placed in muscles, tendons, and joints). Every nervous or sensory organ provides information to the brain. Thus, if children are having difficulty in mastering the letters or numbers - their shape or orientation, then one way therapists can help the child is by writing those numbers on the back of the child's hands. This will be of great assistance in supplementing the information that the child has received through auditory and visual channels. The remedial activities could also include tracing letters and numbers by fingers or writing the numbers and letters in sand, salt or water.

Children's games and play activities that call upon the recognition of letters, numerals, and geometrical shapes on a regular basis can serve a great purpose in remediation.

Games and play activities that involve spatial coordination, orientation and organization, classification and subclassification activities, help children to strengthen their overall spatial and visual perception.

In our work with children, particularly, the learning disabled, we have seen that the children who make mistakes in number concepts, consistently use finger counting, or employ other inefficient methods of counting and have no or a very poor visual image of numbers. At the same time when we present numbers through specific elemental groupings (standard visual clusters and patterns), the children are able to grasp the number concepts much faster and get away from sequential (one-to-one) counting quite easily. Once a child acquires the number concepts in the standard visual cluster form we vary the arrangements to irregular patterns to bring flexibility in visual clustering. The child's ability to develop these visual images of clusters of objects is essential to his understanding of cardinal numbers one through nine. Dunlop and Brennan (1981) state:

> One way we facilitate the development of mental images of quantities represented by the numbers is to provide visual mathematical illustrations which enable children to readily perceive the cardinality of sets containing less than ten objects, i.e. identify and interpret the number of objects in a set without having to count.

Most children by the age of four are able to recognize the clusters involving one, two, three and four objects. At this level the child does not need to count the objects in the cluster, he visually partitions them and makes unconscious correspondence (matching of the cardinality between the collection he is observing and the corresponding image of a standard collection with that cardinality he holds in his visual image memory bank). This matching is immediate and depends on the child's experience with concrete objects and visual clustering.

H. SPECIAL SITUATIONS

Apart from what we have discussed above, there are individual situations and handicaps (visual, auditory, and tactile problems) many children seem to have that create problems with perception and therefore reversal problems and therefore difficulty with number concepts. In those situations the teacher has to try special methods. For example:

- More physical experience before visual representation and auditory recognition of numbers;
- Writing a number on the back of their hands and asking them to match the number of objects with the numeral that was written;
- Cutting numbers from objects of different textures -smooth, rough, etc.
- Making recognizable grooves in wooden cubes;

- Sand water table experience (writing numbers in sand and water);
- Games and other real life activities.

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(a) Help the children to retain the mathematical skills they have acquired during the academic year.

(b) Help the children to remain engaged in the academic learning process during the summer months.

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Program

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The enrollment fee is $50 per student. Special fee arrangements may be made when there is more than one participant from the same family. All materials are mailed to the parents directly. At the end of summer, each child is sent a certificate on completion.

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HOW TO TAKE A CHILD FROM CONCRETE TO ABSTRACT

(MAHESH C. SHARMA)

Early mathematics experiences using concrete materials form the foundation for all later learning in mathematics. A child's early attitudes about mathematics determine his/her interest and motivation towards learning and using mathematics. Without a sufficient number of appropriate concrete mathematical experiences that provide sufficient modelling for mathematical concepts, at the early stage in the child's development, the child will encounter needless learning difficulties in the future. In the early stages of a child's formal learning care must be taken that these mathematical experiences using concrete materials are carefully planned, because experiences that are not appropriate for developing proper skills may create frustration and may also create a negative attitude towards mathematics.

A. Important Factors in Mathematics Learning

Strong and appropriate connections between the learning experiences at different levels must be made. Clear interrelations between the outcomes contemplated by the teacher and the concepts to be acquired by the children must be made. When a child is learning mathematics concepts four vital components are involved simultaneously. These are a child's: **level of cognitive development, mathematics learning style, preparation in pre-requisite and support skills,** and **emotional maturity.** A breakdown in any one or more of these factors could be a reason for a child's mathematics learning problems and his difficulties in using and applying mathematics.

Apart from all of the above considerations that relate particularly to the child, another factor that is also important is the teacher. A teacher needs to provide an exciting and challenging learning environment. She needs to plan instructional activities using a diversity of instructional materials. A teacher's style of presentation of mathematical instruction plays an important role in a child's ability to conceptualize mathematical concepts. Learning experiences and the environment should reflect all of the above considerations to the maximum possible level. Such an environment, of course, should include opportunities for the child to learn from interacting with other children and adults that come in contact with him/her during formal and informal instruction.

Mathematical experiences must be very concrete and must be appropriate to the child's developmental level. Many teachers have the tendency to rush mathematical experiences from the concrete level to the abstract level. Many teachers provide a few concrete experiences related to a particular concept and they think that the child does not need any more concrete experiences. Generally, when a child has understood and conceptualized a concept and automatized the skill involved in the concept, he will outgrow the concrete material relating to the concept. This means he has acquired the concrete model for that concept and can visualize the model when needed. If the use of the concrete material is curtailed before the mastery is reached, then either the child will have difficulty in mastering the concept or will rely and develop inefficient strategies and methods. Another thing that teachers should keep in mind is that the demonstrations using concrete materials are poor substitutes for actual concrete manipulation of objects by the child himself. Very young children are pushed to count, to write numerals, to add etc. without appropriate concrete activities. These abstract activities, introduced before the child is actually ready to intellectually appreciate and even understand effectively, do the child more harm than good. This is particularly so, if the child has not experienced many concrete readiness activities involving manipulative aids and pictures prior to working with abstract symbols. Part of the difficulty that many children have with mathematics concepts is due to its abstract nature. Mathematics experience could become more enjoyable if these numerical concepts were presented in tangible, concrete exercises which relate to the individual child's needs or interests. This does not mean a child should not be introduced to abstract concepts at an early age. Of course, he should be. But, the important thing to remember is that it should be done properly and one should not push for mastery by rote rather it should be done by providing appropriate concrete models.

B. Need For Manipulative Materials

Almost all teachers have experienced the following situation: you ask a child a simple question involving addition or subtraction facts. You look at the child. There does not seem to be any reaction on the child's face; he simply stares at you. You repeat the question; you change the language of the question; you explain the relevant information to the question. You are in the midst of this process and all of a sudden the child blurts out the answer. Naturally, you wonder what had happened. But, you might have not noticed one thing. During all this time that you were explaining the question, the child was busy calculating the answer on his fingers behind his back or under the table. He was using a concrete (using fingers for counting) model for calculating the appropriate computation. Using fingers is one of the concrete models that children continue to use even when these models are inappropriate and inefficient. But here our point is not to discuss whether the use of fingers is appropriate or not, we want to point out that it indicates that the child is demonstrating a partial level of mastery. This should not be interpreted to mean that using fingers for counting is bad (some approve of it, others don't). One thing it does indicate is that the child may still need concrete, manipulative materials to help conceptualize mathematical ideas. The use of fingers is not a cause for the child's mathematical problems; it is only one of the symptoms of such a problem. Moreover, using a finger to represent a number is a poor use of fingers. As soon as one has counted up to ten, one has run out of fingers and therefore has difficulty in keeping the tally if one has to continue the count.

Using fingers could be made an efficient activity if one decides that it is an effective way of counting. In our opinion use of fingers for counting is not an efficient method for computing. There are excellent methods that use fingers effectively for counting. The most popular method for finger use in calculation is "Chisonbop" or "Fingermath". In this method specific values are assigned to fingers in both hands and they are used according to predetermined rules for calculations. In several eastern countries, fingers are used in very creative ways for calculations and counting. It is also true that in some cases, particulary in the case of those children who are kinesthetically oriented, the use of fingers may be the most effective way of learning arithmetic. In most other cases, as soon as the child has been provided with appropriate concrete manipulative experiences and effective ways of their use the child learns mathematics much more quickly and effectively. As a result of these concrete experiences he is able to realize that there are better counting strategies and usually drops the use of fingers automatically. By force and nagging it is impossible to eliminate the habit of calculations using fingers. As long as the child is not presented with more effective methods, it is difficult for the child to drop this ineffective method of counting. Only in a small percent of cases have we found that even with the help of extensive work with manipulative materials have children not been able to drop the use of fingers for calculations. It is, generally, the case with other children who are still using fingers at age 14, 15, or later. To develop new models for counting and calculations using concrete materials or thinking strategies, in such cases, are some what difficult because one has to replace an inefficient strategy for an efficient one. But, we have found that persistance in changing the habit and inculcating thinking strategies pays generous dividends.

C. How Long? How Much?

Another common complaint from teachers in the upper elementery grades is that many children in their classrooms have difficulty in performing computations without the use of manipulative materials — cuisennaire rods, unifix cubes, chips, blocks, etc. (Manipulative materials are not commonly used in the upper elementary and junior high school classes and as a result many of these children are poor in those mathematics concepts where computation is the basis.) When they consult with the children's previous teachers they are informed that these children were doing fine in their classrooms. Of course the teachers from earlier grades forget to report that the children's mastery of facts was limited to giving answers with the aid of concrete materials. Their mastery was reached only to the concrete level of knowing rather than the abstract and communication level of knowing. And this partial level of mastery was misjudged by the teacher for complete mastery. This raises important questions:

> "How long should a teacher continue to use concrete materials in teaching mathematics concepts? How does one decide when to curtail the use of concrete materials in teaching mathematics?"
>
> "Why do some children seem to be lost when manipulative materials are taken away from them? Or why is it that abstract ideas do not have any or limited meaning to many children in the upper elementary grades?"
>
> "When and how to remove munipulative materials as instructional aids in the teaching of mathematics concepts?"

Of course, there is no one answer for every situation. It has to be determined in each individual case, but some general guide lines can be considered. How to take a child from concrete to abstract is the focus of the rest of this article.

The use of concrete materials such as counting blocks in solving problems particularly word problems has been examined, and it has been found that the availability of counting blocks (e.g., Bolduc, 1970; Hebbeler, 1977; LeBlanc, 1968; Steffe & Johnson, 1971) and/or reference dolls or pictures (e.g. Harvey, 1976; Ibarra & Lindvall, 1979; Marshall, 1976; Shores & Underhill, 1976) facilitate the solution of problems, particularly for young children. In most studies they found that the use of concrete materials provides general improvement in children's problem solving abilities. The most important aspect of the use of concrete materials is that it motivates children to solve problems. For example, Carpenter, Hiebert, and Moser (1981) found that when first grade children were given the choice to solve with or without concrete materials, they preferred to solve problems with concrete materials. The use of concrete materials by the child is definitely beneficial but even observing the use of concrete materials has some positive effects on children's problem solving abilities (e.g., Buckingham & MacLatchy, 1930; Gibb, 1956; Ibarra & Lindvall, 1979).

D. Mathematics Concept-triad

Every mathematical concept consists of three components: *linguistic, conceptual* (idea), and *skill*. The linguistic component deals with the language of mathematics and to develop this the teacher has to teach mathematics as a language. The conceptual component deals with the core of that mathematical concept. To conceptualize the inherent idea, the child needs a model for it and at introductory teaching level (when a child is learning it for the first time), he needs concrete materials to form the model of the concept. If the child does not have the model, the concept eludes him. For example, when you ask a child the following question:

> "I have 17 x 13 in my hand. I want to make it 17 x 14. What should I add to it?"

If the child answers: add 1 or 14, then it shows that he has not developed the conceptual model for the concept of multiplication. He needs concrete experiences (in this case the use of Cuisenaire rods will be a more appropriate material rather than counting a set of things) to form the model. For detailed discussion of the development of multiplication concept and other specific concept refer to back issues of this publication.

The skill component deals with the skill or the algorithm associated with the concept. The skills are developed by reinforcement and practice. The first step in the development of a skill is to do task analysis of the skill and then practice the component tasks. The student needs to be observed to see whether he is practicing the skill correctly or not. And therefore, the oral mathematics or mental arithmetic exercises are the most effective means of developing the skills component of mathematics learning.

It seems that for many children, who need to use concrete materials extensively, abstractions or even semi-abstractions (visual/pictorial representations) are difficult to comprehend without concrete models of the abstract concepts. Manipulative materials for them are not an alternative mode of learning. They are a necessity. The key is how to use the concrete materials effectively and how to take a child from concrete to abstract. It is not the manipulative materials that teach, it is the teacher who teaches. Every manipulative material is a static piece of equipment and only in the hands of a competent teacher does it acquire the dynamism which is responsible for helping children learn. Similarly, no one particular material, however good it may be, is appropriate for every child or even for all arithmetic concepts. Each child has a preference and a natural affinity for one kind of material over another. Mastery in the use of a particular instructional material comes with use; and with use and practice any instructional material could become versatile. It is not the material that is being taught. It is mathematics that is being taught through the use of a particular instructional material to a child who is growing, developing and changing with time and learning. Instructional materials by their virtue and use may greatly simplify the learning process for the child and the teaching process for the teacher. A teacher does not select the instructional material just because it is going to help the child learn that particular concept but many times the child's use of instructional material helps the teacher understand the learning process. The selection of a particular type of instructional material and its manipulation by the child provides a great deal of information regarding his learning process and his strengths and weaknesses.

E. Key Skills

Children continue to use manipulative materials for several reasons. First, most children up to the fourth grade level (chronological age approximately 10 years) are still in the concrete operational stage of cognitive development where major means of conceptualizations are through concrete manipulations. (Piaget, a Swiss psychologist, has formulated four major stages of a child's intellectual development beginning with the sensory motor stage to the formal operational stage.) During the concrete operational stage of development, children learn and conceptualize through concrete manipulative experiences. They need "hands on" activities and experiences before they can learn abstract ideas and concepts. In most cases these "hands on" activities become the basis for generalizations and abstractions. The use of concrete materials provide them with the security of working at their own pace and with the opportunity of breaking the concept into manageable pieces.

According to Piaget, even if a child is in the formal operations stage, when he encounters a new or a complex concept (for example a secondary concept — a concept that is formed by combining several primary concepts and operations), or operation, he goes back to the strategies that he used during the sensory, pre-operational and concrete operational stage. He attacks the new problem from a lower vantage point of view. He utilizes the strategies from his previous cognitive stage of cognitive development. Although he may stay on the lower stage only for a very short while. To effectively implement these strategies the child needs concrete materials and models. In the absence of these concrete materials and models, the student has to improvise the concrete models and which may or may not be possible for him. In most cases this improvisation takes the form of using fingers inefficiently and other things of a similar lack of sophistication.

A second reason that children remain "hooked on" to manipulative materials might be that often a teacher uses only one type of manipulative materials for the introduction, mastery reinforcement and evaluation of a concept. Thus, to many children the use of that particular set of materials becomes an end in itself rather than the means for arriving at a solution or learning of a concept. They feel that if they do not use those particular materials as a means of solving the problem then they are not doing it right. In other words these children become "material specific." To them that particular concrete material becomes identified with a particular mathematical concept. When they encounter that particular concept they think of that particular material only. To avoid this a teacher needs to present a mathematical concept in more than one concrete material and model.

Another reason might be that the children, even with the use of materials, have not developed the three main prerequisite concepts that are extremely important in the process of learning numbers and operations with them. Namely, (a) the recognition of **visual clusters** in the arrangement of numbers, (b) the **estimation** of the cardinality of a set and (c) the estimation of the outcome of an operation.

F. Perception and Mathematics Learning

The very nature of mathematics insists that a person must first perceive objects, their attributes and their relationships. The earlier mathematical concepts, particularly primary concepts, (whether by a child or an adult) are learned through senses (Skemp, 1963). Mathematical information is first received by the learner's senses, then perceived by the senses and then sent to his appropriate processing mechanism in his neuropsychological system.

Several studies have shown a positive correlation between spatial ability and achievement in mathematics (Sherman, 1980; Tarnapol and Tarnapol 1979; Fennema and Sherman, 1977; Guay and McDaniel, 1977; McGannon, 1977; Smith, 1964). In one such study Fleischner and Frank show visual spatial ability to be a better predictor than I.Q.

for success in mathematics (Fleischner and Frank, 1979). Developmental psychologists such as Skemp, Piaget, and Bruner also hypothesize that the mathematics concept aquisition is dependent on the ability to perceive relationships visually and generalize them into logical inferences from these perceptions. Piaget adds that sensory motor activities prepare the child for logical operations. Logic, in the earlier stages when the language development is taking place, is based on the general coordination of actions and later as the interaction between the language and actions both concrete and mental/intellectual. Basic then to the learning of mathematics, at the earliest level, is perception, the process of organizing the raw data obtained through the senses and interpreting its meaning through the perceptual and logical operations in the brain. Therefore, a student who has poor visual/spatial ability may have difficulty in mathematics. This is for two reasons: First, the student does not understand mathematical concepts because he has difficulty in visualizing the relationships on which they are based; second, the student has difficulty munipulating the symbols mentally and later on paper.

Piaget considers the interplay of perceptions or sensations as the ingredients for the formation of intelligence. The cornerstone of this process is obtaining and organizing information through the senses. Therefore, a student who has poor visual spatial ability may have difficulty in mathematics. The use of manipulative instructional materials helps a child in developing some of these visual/spatial abilities.

It is not just the use of concrete materials that improves mathematical understanding, but the explicit link between meaningful actions on the objects and then its translation into related symbols. Thus, one has to focus on the whole sequence of the conceptualization process. The basic sensory modalities through which mathematical information passes through are: visual, auditory, tactile/kinesthetic. In other words, every concrete munipulation should be recorded and if possible abstraction and generalization made right after or during the concrete manipulative experience.

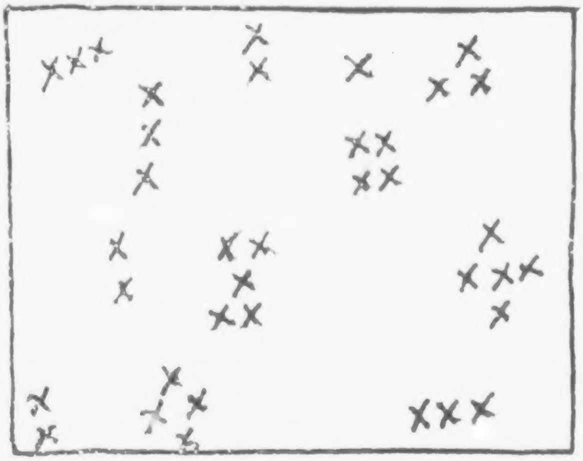
Linking the meaningful actions on the objects to the related symbols is accomplished through mentally arranging, rearranging experiencing, abstracting meaningful relationships, making generalizations, and consciously and unconsciously comparing every new experience with what one already possesses, but most importantly that the student realizes that mathematics is a natural way of expressing concrete relationships. The student experiences these concrete relationships through an enactive experience, then he constructs a symbol to record the experience and attaches language to it (Van de Walle, 1983; Hamrick, 1979; Bruner, 1966). Most times this is the weakest link in the sequence of teaching and learning of mathematics.

G. Learning Process

Although, every new mathematics concept has its own cognitive complexity but at the same time it is dependent on many of the earlier concepts because each concept is developmental and cumulative in nature. Formation and conceptualization of every new mathematical concept is dependent on one's earlier knowledge and skills. For example multiplication is dependent on addition, solving quadratic equations on algebraic manipulations such as factoring and collecting of like terms, they in turn being dependent on the commutative and distributive laws and other foundations of algebra. For the mastery of any mathematical content, therefore, we need both kinds of information: rote and schematic. When there is a schematic dependence between concepts and processes, the presence of prerequisite schemas is a must. Where the appropriate schemas for the learning of a specific content are absent, some kind of learning is possible. But without the knowledge which is necessary to give meaning to the new material it will be learning without understanding. This latter kind is isolated learning, whereas, in schematic learnings new knowledge builds upon the previous knowledge and on existing schemas. In schematic conceptual learning new schemas are formed and old ones are restructured and refined into new schemas and, therefore, learning. Retention in rote learning is short lived and selective, whereas in schematic conceptual learning the learned material can be used after a long time. The intact and efficient memory system, therefore, is an essential requirement for optimal functioning in mathematics.

H. Visual Clustering

Let us consider the following example: Arrange a set of objects in a tray as shown:



(Figure 1: Arrangement of objects on a tray.)

Now ask the child to count the number of objects on the tray. You will see children use two major strategies or a combination thereof. In one, the child might count the objects one by one and give the final answer as to how many objects are in the tray. The process could be represented as follows:

$$1 + 1 + 1 + \ldots\ldots\ldots\ldots\ldots + 1 = 37.$$

Another child might count the objects in clusters. The outcome of this process could be summarized as follows:

$$3 + 3 + 2 + 2 + 3 + 4 + 5 + 3 + 4 + 5 + 1 + 2 = 37.$$

The first child has not yet understood the concept of visual clustering and needs many more concrete experiences before going on to operations with numbers, and even then he will need more concrete experiences. Whereas, the second child has understood the concept of visual clustering and will soon discard the use of manipulative materials for counting. He is able to form visual pictures of clusters of objects and is able to represent numbers in the form of clusters and thus will have greater facility in the operation of numbers.

Visual clustering is the ability to identify the number of objects in a collection by visually approximating it, which means counting in clusters rather than one-to-one; as the second child did in the above example. A child must be able to perceive the number of objects (up to 7 + or − 2) in a group as a result of observing their configuration and without counting them. Through regrouping and munipulating objects mentally and comparing them with frequently presented regular patterns (such as those on a die) the child is then able to organize (visualize in his/her mind) those objects that are in a different array so that s/he knows the amount without counting. This eventually leads to integration of the collection of objects, its cardinality, and the shape of the numeral representing the collection. Once the child is able to integrate, (a) the visual arrangement of the collection, (b) the shape of the numeral, and (c) the associated sound to the numeral s/he has conceptualized the number. This means that as soon as s/he sees the visual display of the objects the number comes to his/her mind immediately and when s/he sees the number the cluster and the shape of the numeral flashes in her mind. Visual clustering is the key in taking the child away from fingers and one-to-one sequential counting.

The best concrete material for developing visual clustering is a regular deck of playing cards. The display on the playing cards are arranged in "regular" clusters, and when the child sees these regular clusters he often brings to form visual images in his mind. Once that image is stabilized in his mind, he can intergrate the cluster, the quantity, and the representation of the number. Once this integration is achieved the child no longer has to do sequential counting, which is a very primitive and inefficient way of counting. [We have developed several games using playing cards to reinforce basic arithmetic skills and facts. Children thoroughly enjoy them and as a result these games form an integral part of our remedial and reinforcement activities at the Center.]

Chart 1

1	11	21	31	41	51	61	71	81	91	101	111
2	12	22	32	42	52	62	72	82	92		
3	13	23	33	43	53	63	73	83	93		
4	14	24	34	44	54	64	74	84	94		
5	15	25	35	45	55	65	75	85	95		115
6	16	26	36	46	56	66	76	86	96		
7	17	27	37	47	57	67	77	87	97		
8	18	28	38	48	58	68	78	88	98		
9	19	29	39	49	59	69	79	89	99		119
10	20	30	40	50	60	70	80	90	100	110	120

Chart 2

1	11	21	31			61			91	101	111
2											
3											
4				44		64					
5	15	25			55				95		115
6						66	76				
7											
8					58			88			
9											119
10	20	30	40	50	60	70	80	90	100		

In the early grades there is much emphasis placed on rote, and sequential counting. For example, most children while counting sequentially do not pay any attention to the change that takes place in the representation of number when moves from 9 to 10 to 11 are made. They are not able to visualize that change; and as a result they treat these three numbers just as any three consecutive and similar numbers in the same way as he sees 5, 6, and 7. This change, from 9 to 10 to 11, from 19 to 20 to 21, etc. is an important change and the child either should discover it by himself or the teacher should carefully make the child aware of it. The best way to do this is to have a number grid (see chart 1) in the classroom at several places along with the sequential display of the number line. There should be one chart (see chart 2) on the chalk-board that has few important entries on it. Kindergarten, first, and second grade teachers should take a few minutes of their class time each day to ask each student to fill some of the entries on the chart. This exercise is particularly important for children in kindergarten through fourth grade. And in the case of children who are having difficulty in learning number concepts even in the higher grades for them this activity should be used in resource rooms.

Activities relating to number grid:

Step 1: Ask the child to identify numbers on the grid at random. Point to an empty space and ask him to identify the number that should occupy that place.

Step 2: Ask the child to read rows: 1, 11, 21,......, 91, 101,...; 7, 17, 27,....,57,....,107, 117,.....; etc.

Step 3: Ask the child to read rows backwards: 98, 88,....,48,....,18, 8; 76, 66,....,26, 16, 6; etc.

Step 4: Ask questions such as 15 + 10 = ?, 77 + 10 = ? and let the child discover the pattern that answer is the entry in the place adjacent in the next row. If the child has difficulty in discovering the pattern, help him discover it by asking more questions and giving him some clues. But do not give the pattern without this process.

Step 5: Ask questions such as 15 + 11 = ?, 77 + 11 = ? and let the child discover the pattern that he can find the answer easily by first asking questions such as 15 + 10 = ?, 77 + 10 = ?, etc and then adding one more. Once again if the child has difficulty in discovering the pattern ask more questions and give him clues.

Step 6: Ask questions such as 9 + 5 = ?, 19 + 5 = ? 39 + 5 = ?, etc. Ask him to discover the pattern. You should ask dozens of questions like these and extend the pattern to more complex related facts.

Step 7: Ask questions as 77 − 10 = ?, 56 − 10 = ?, And follow the pattern as in steps 3, 4, 5, and 6 by replacing the addition symbol by the subtraction symbol.

An important thing to remember in using the grid is to avoid sequential counting and help children discover patterns between number relationships and to form visual image of the number grid so that children can see number locations and the relationships between them easily.

I. Estimation

Estimation is the ability to see the range of the possible answer(s). In real life, when we solve problems we estimate more than 90% of the time compared to less than 10% of the time when we perform actual calculations. This is just the opposite of school situations where we perform actual calculations more than 90% of the time and estimations only about 5% of the time. "Estimation is a rough solution to a problem in counting and measurement: it starts with a real-world referent and ends in descriptive numbers. We hypothesize at least two fundamental types of estimation techniques — benchmark and decomposition/recomposition — and argue that these are hierarchically organized temporally sequential, and developmentally oriented" (Siegel, 1978) The development of estimation (as a cognitive ability) is a constructive and developmental process: the representation of events and things is initially concrete and action-based, then imagic-visual, figurative, and intuitive, and finally, verbal-linguistic-symbolic and "operative" (Piaget, 1971).

One needs to distinguish here between estimation and approximation: approximation is a rough solution to an arithmetic or basically a numerical problem that starts and ends in numbers. Approximation indicates the presence of the underlying algorithms and sequential procedures. For example, to say that 59 x 61 is roughly 3600 is an approximation and shows that the person knows the concept and the process of multiplication. Estimation on the other hand is an ability that is essential for conceptualization of ideas; it is used in modeling the environment. As when young children play baseball and one child asks the other to throw the ball a little slower and a little higher. The child in response to this does not make any calculations, he estimates. He adjusts the physical model governing this concept in his mind, and responds accordingly. It is an iterative process. Estimation begins with modeling, with the representational activity. This manifests later on in numerical approximations. Thus approximation helps to facilitate calculations and brings efficiency in solving numerical problems, while estimation helps the student to gain reasonably accurate cognitive models (representations) of the world around us.

> The necessity of the existence of a quantification operator other than subitizing and counting is clear since they only deal with discontinuous quanity... The importance of the estimation quantification operator arises from its contribution to repairing the deficiencies of subitizing and counting (Klhar Wallace, 1976, p. 64).

In computation and also in solving word problems, people first construct an internal presentation (visual, spatial or motoric) of the terms, numbers, and operations, and subsequently transform this internal representation in to formal equations or other mathematical expressions (Larkin, 1977).

It is the visual-figurative estimation that is the key in understanding not only early number concepts but even later problem solving activities. Still, in our teaching and research we have paid more attention to the linguistic verbal knowledge system which favors children with better sequential, deductive left hemispheric processing. Only recently have psychologists and others begun to investigate the development and utilization of the visual spatial knowledge system.

The cognitive ability to estimate is difficult to diagnose. It can only be done with the help of a combination of concrete experiences, keen observations of the child's work and interviewing the child while he is performing the operations on concrete materials. It involves two different processes: benchmark estimation, and decomposition/recomposition. Benchmark estimation is the simpler of the two, and is merely a comparison with a known standard: one finds a suitable benchmark (criterion), compares the selected benchmark to the to-be-estimated item, and checks to see if the answer is reasonable. In the decomposition process, one breaks down the problem into manageable sizes and applies benchmark estimation to each component problem and then recomposes the component estimates into a total estimate and checks the reasonableness of the estimate. Sometimes one decomposition may not be enough, in that case a series of successive decompositions may be necessary. In each of these processes the more experiences the child has the better prepared he is for the right kind of benchmark and the right type of decompositions of the given problem.

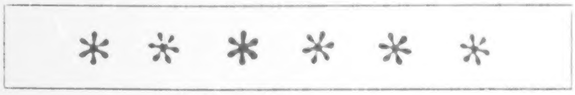
A child who is a good estimator is willing to try more than one approach to solve a problem because he can quickly estimate the solution using the estimation process and therefore test the validity of the approach that he has planned to take in this particular situation. A poor estimator has to go through the whole algorithm and even then he may not be sure about his answer for that problem. For example, a poor estimator will multiply 12 x 14 and may give the answer 18 and not be aware that s/he is wrong. Before any actual arithmetic calculations are actually performed a teacher or parent should help the child estimate the outcome of the problem. Our observations show that good estimators are better problem solvers. It seems that it works because better estimators check both the process involved in the problem and the outcome of that process-the solution.

Before the child begins working with mathematical operations, s/he needs to have an approximate sense of change in quantity or shape when objects are acted upon. Beginning with non-verbal, manipulative experiences, the child develops the ability to predict a plausible outcome. This ability not only applies to numerical outcomes but also to non-numerical situations. For example, when two triangles are joined, s/he may predict that they could assume a diamond or a parallelogram shape, but not be circular. The skill of estimation progresses through ordering and combining sets so that the child can state that these are more than, less than or the same after some change in quantity or position of the sets. This will eventually lead the child to a sense of reasonableness of his/her answer when performing operations. Working with questions on estimation gives the child a greater feeling for numbers and relationships and therefore increases his/her competence in math.

J. Tactile And Logico - Mathematical Experiences

A child's success in either of these concepts — visual clustering and estimation, is dependent on the extent of the child's concrete experiences. Thus, to help children in the abstract mathematical activities, the teacher needs to assess the child's level of cognitive development and the extent of concrete experiences he has had. Many teachers at the primary grade levels do provide extensive concrete experiences even though they feel that they will make children dependent on manipulative materials and later in the higher grades when they find children helpless without the concrete materials they completely abandon the use of concrete materials. In such instances, the concrete materials and experiences are not to be blamed. As mentioned earlier the recording of the concrete experience must follow immediately after the actual concrete experience. A teacher is on the right path when s/he realizes the importance of concrete experiences in introducing a new mathematical concept or for reinforcing an old one. However, it's the proper use of concrete materials and the extraction of the appropriate concept idea that matters.

Every physical (concrete manipulative experiment) experience consists of two important components: the *tactile* activity and *logico-mathematical* experience. For example, when a child is counting a collection of flowers by touching each one, he is involved in a tactile activity.



But, when he finds out that the number of flowers remains six (see above figure) whether he counts the flowers from left to right or right to left then he is involved in a logico-mathematical experience.

Another example would be to assume that a child is playing with a scale. He places unequal weights in the two pans of the scale, as a result the scale is now in an unbalanced position with the heavier side down and the lighter pan up. So far the child has performed a tactile activity but when he takes out some weights from the heavier pan or places some some weights away from the heavier pan and adding some to the lighter pan and is able to balance the scale he has acquired a logico-mathematical experience. These types of logico-mathematical experiences become the basis of future mathematical and scientific concepts. For example, the physical experience of playing with a see-saw provides a concrete model for the concept of the law of moments in physics — which is defined as:

> The sum of the products of the weights and their distances from the center is equal to zero.

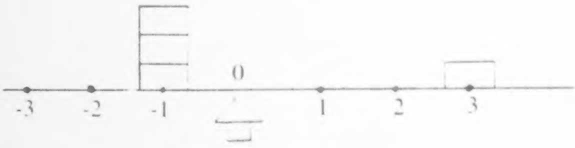


FIGURE 3

The ability to perform Piagetian tasks, such as conservation, have been closely associated with achievement in mathematics (Taloumus 1979, Guay 1977). This can be attributed to the fact there is a significant relationship between Piagetian tasks and spatial ability (Williams and Brekke 1979) and when children are involved with concrete experience, they gain logico-mathematical experience which in turn result in cognitive abilities.

Many children have difficulty with abstract concepts in mathematics, and problems in abstract mathematical conceptualization arise because many children are not able to operate on the abstract, formal and logical levels of mathematics as they have limited appropriate and corresponding physical/concrete/visual experiences involving both tactile and logico-mathematical components. There are three possible situations which result in limited logico-mathematical experiences.

(a) A teacher does not provide any or limited concrete experiences, therefore, there is a limited or no possibility of logico-mathematical discoveries. Similarly, when children have no or few toys and games to play at home, they have limited opportunities of developing logico-mathematical experiences.

(b) A teacher provides concrete experiences but no opportunity is provided for children to derive logico-mathematical deductions from their concrete manipulative experiences. The activity remains at the physical tactile level only and no discussion of that activity takes place in order to abstract logico-mathematical skills.

(c) A teacher provides enough concrete experiences, but when the opportunity for logico-mathematical deductions from those concrete experiences arrives she provides the logico-mathematical deductions herself and explains all the possible inherent options and patterns from that experience. This deprives the children of the essential growth that was possible from the deductions from that experience.

Thus, it is not the concrete experiences only that need to be emphasized but also the concomittant logico-mathematical component of the experiences. As a matter of fact, whenever I emphasize the importance of concrete experiences in my lectures and courses, many teachers say that they have tried all the possible materials and concrete experiences in their classes and the children are still having difficulties in conceptualizing, learning, and applying mathematics ideas. When I probe further I generally find that: **first,** in most cases the concrete experience consists of demonstrating the use of a particular material and from this demonstration they expect then that the demonstration will be enough for the children to help them understand the related mathematical idea; **second,** in other cases they actually use the materials but the time alloted to the logico-mathematical component of the experience is very limited. As a result, children are not able to derive the maximum benefit from the use of the concrete experiences. For example, in "counting of the flower experiment" that we cited earlier, a teacher might say, "children if you count these flowers...... yes! you are right there are six of them. Now if you count them from left to right and then count them from right to left and you will find that the number of flowers, you find is still six." This interaction between the children and the teacher is an instructive example. As you notice, the teacher did provide a concrete experience for the children but she provided the logico-mathematical experience also by saying, "you will find that the number of flowers is still six." This teacher should have provided some time for the children to explore and to see that they could come up with that conclusion. In case the children did not arrive at that conclusion then the teacher should try to guide their explorations or provide more concrete experiences. To provide vicarious experiences robs children of the pleasures of discovery and the ability to explore and make conclusions, which is an important part in the learning of a concept and for transiting from concrete to abstract. It is not enough to provide concrete experiences. Every concrete experience must accompany "guided discovery." This "guided discovery" process helps children to make proper use of concrete manipulative experiences and helps them in development of logic-mathematical experiences.

K. The Five Step Spiral

Step 1: **Free Play**

A teacher should introduce every new concept through manipulative materials and concrete experiences. And even before the teacher introduces the concept proper, she should allow the children some time for free play and exploration (even if it is only for few minutes). This provides them with an opportunity to indentify with the materials and a chance to satisfy their natural urge of sensory exploration and tactile input. If this step is bypassed then the teacher will be paying a high price in terms of distractions that children will create later in order to satisfy their natural curiosity when these materials will be used with more focussed purpose and time. This period of free play allows them to express themselves freely and initiates the manifestation of the creative instincts through concrete materials. The period of free activity with materials is important not only from the children's perspective but even we adults when confronted with a new concept resort to the sensory interaction level and try to derive information about that concept through concrete means, by conducting an experiment, by setting up or concocting an example or a model for that concept. All of these are examples of sensory involvement in the learning process. For example, when we buy a new object, even a book, the first level evaluation (critique) of the object is dependent upon the sensory data that we collect by sensory examination of that object: how does the object feel, look, or smell to the senses. The free play activity also provides the opportunity for exploration, discovery, logico-mathematical experiences and the strengthening of visuo-spatial abilities. The instructional materials themselves are static in nature but the teacher and her approach to teaching makes them dynamic; inclusion of guided and limited free play with materials is the beginning of the process that turns them from static to dynamic learning instruments in the learning process.

Step 2: **Primary Operations and Primary Concepts**

After the teacher has provided the children time for "free play" and interaction time with the materials, she should identify all the primary concepts involved in the problem and draw the children's attention to the primary concepts involved in the concept under consideration. This means that the first few minutes of time after the free play activity should be devoted on the primary concepts involved. If the concept, that the teacher is teaching now is a primary concept then the teacher should devote sometime to those earlier concepts that are related to the concept and those concepts that are the prerequisite and support skills related to the concepts being taught. The importance of prerequisite skills can not be over emphasized (Sharma, 1980). For example, let us assume that the teacher wants to introduce the multiplication algorithm:

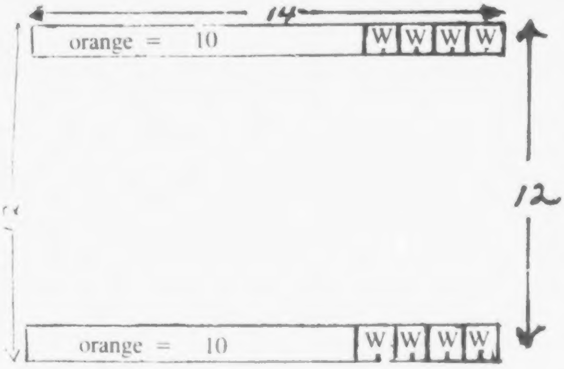
$$\begin{array}{r} 14 \\ \times\ 12 \end{array}$$

This is a secondary concept because it is made up of several primary concepts. The primary concepts involved in this problem are: multiplication facts, place value, addition, and following sequential directions. Following directions is a pre-skill for this concept. The ability to follow sequential directions is a crucial one. As a preparation for the multiplication algorithm, therefore, it is necessary that a teacher devotes a few minutes on each one of the primary concepts and pre-requisite skills. This should have two major purposes: reinforcing the earlier concepts and setting the scene for the new concept. This step not only helps the child to consolidate and reinforce the previous learning but also provides a chance to see the interrelationships between different concepts and procedures. This is also an opportunity for the teacher to show the emerging patterns in the process.

Since in every classroom, children are functioning at different cognitive and academic performance levels, it is important that the primary concepts and the pre-requisite skill activities are presented at different levels of difficulty, mastery and understanding. The optimal way of doing this is through multi-group activities. The teacher should select several activities — board games, card games, math fact games, commercial games to reinforce concepts and math facts, projects that involve applications of concepts, planning of mathematics content between two or more children and independent board and seat work — these should be happening at the same time in the classroom. This type of multiactivity instruction in the classroom requires careful planning and a high degree of competence on the part of the teacher. It involves management techniques and record keeping procedures. Above all it requires time, creativity and a teacher who is genuinely interested in the individual development and achievement of her students. The multiactivity instruction definitely changes the organization of the classroom. The teacher becomes a learner, researcher, and planner in the true sense of the term. This teacher constantly draws suggestions from research and established practice that can help to provide for all children with different cognitive levels.

Step 3: **Manipulative Activities**

The teacher should now devote time on the concept under consideration through manipulative material and "hands on" activities. Considering the above example, the teacher could use the Cuisennaire rods in the following arrangements.



(Fig. 4: Arrangements of Cuisennaire rods on the table.)
W - white = 1

Figure 4

The child counts the 12 orange rods as 120. 12 groups of 4 white rods = 48. The total outcome is 120 + 48 = 168.

The most important factor in this step of the process is that the teacher should use a variety of materials. The variety should reflect two important considerations:

(1) If a teacher uses only one type of material in the presentation of most of the mathematics concepts, a child could become "material specific" — i.e., the child might think that the concept is dependent on that particular material. Whenever, a child is asked to solve any addition problem he might draw a number line, whether one was needed or not. On questioning, it becomes clear that teacher, in this child's case, always used a number line to explain arithmetic facts. The child, therefore, associates the number line with the arithmetic problems and the solution process for him is incomplete without drawing the number line. We have seen similar examples even with older children.

(2) We have observed that some children learn better with the use of "discrete materials." Discrete materials are those instructional materials which have many parts and each component of the materials has equal value and the value is independent of the other parts of the material. The other characteristic of the discrete materials is that the key skill used in this type of material is sequential counting. For example, the number line, counting blocks, fingers, invicta balance, etc. are in the category of discrete, discontinuous, sequential materials. By far these are the most popular and commonly used materials found in American elementary schools. For example, because of the prevelance of sequential materials many children begin to rely very heavily on the use of fingers for sequential counting.

Other children learn better with the use of "continuous materials." Continuous materials are those instructional materials that are dependent on the visual spatial nature of the material. The materials where color, shape, size, pattern, etc. might be the key features. And sequential counting is not emphasized, although it may be a subsidiary feature. In other words, without the use of sequential counting procedures one could assign numerical values. For example, the Cuisennaire rods, chip trading, graph papers, and many of the board games will fall into this category. Some other children learn by using both kinds of materials — discrete and continuous.

It is, therefore, extremely important that a teacher uses a variety of materials. Having only one kind of material in the classroom is as useless (particularly for the child who needs it) as not having any materials. Ideally, it is advisable to use both kinds of materials to represent the mathematics concept being taught. When the teacher uses both types of materials, the children with different needs will be able to utilize the appropriate material for their own learning needs. It is also important to note that no one material is appropriate for all the concepts, for all the children and for all levels of mathematics learning. A good teacher is constantly looking for optimal learning materials. Many times teachers look for content in a particular material i.e., they associate a particular material with a particular concept only. It is limiting not only to the material but also to the teacher. Instead of being creative she becomes follower of a package and all the spontaneity in teaching is gone and the students may lose the enthusiasm for learning.

Step 4: **Recording the Concrete Experience**

Most of the time, the teachers in the primary grades end the instructional activity at the previous step. Therefore, the child's mastery level reached is at the concrete level: he can perform the arithmetic operation at the concrete level. If the concrete activity is not carried to its representational and abstract level, then many of the children will not be able to abstract the inherent mathematics concept from the concrete experience for themselves, and therefore their mastery level of that concept will remain at a lower level. Or just the opposite is present: the teacher presents the concepts only at the abstract level without using any concrete materials and ceptualizing the idea because of thee absence of a concrete model.

In the examples discussed in the previous section, the child has the answer for the problem but it is dependent on the concrete representation and therefore as long as the materials are present the child can do the problem, but s/he has not conceptualized the idea yet. The idea has to be abstracted and generalized irrespective of the concrete representation. The abstraction of the concept will take place by the use of multiple representations and by transferring the concrete material information to pictorial and semi-abstract representation. Research shows that students benefit from exposure to multiple representations of the same concept using different materials and different types of materials, e.g., 12 x 14 can be represented using Cuisennaire rods, use of grids on a piece of graph paper, Dienes blocks, etc. However, children should be encouraged to identify all of the mathematics commonalities represented by the concrete materials and if they fail to see them then the teacher can point them out to the children.

Since learning at this level is so dependent on the perceptual input, and children are relying on this perceptual input, it is important that the teachers help them in seeing and making mathematical generalizations as much as possible, e.g., making sure that they do not associate "triangle" only with right triangle or equilateral triangle. Most children do that because they were shown only specific examples through concrete materials.

Children need to graphically symbolize (record using mathematical symbols) the outcome of the concrete experience. Considering the above example, the recording will take the following form:

$$\begin{array}{r} 12 \\ \times\ 14 \\ \hline \end{array}$$

= 12 (orange rods) + 12 (4 white rods)
= 12 groups of orange rods + 12 groups of 4 white rods
=12 x 10 + 12 x 4 (1 orange rod = 10; 1 white rod = 1) = 120 + 48
= 168

Thus to summarize:

$$\begin{array}{rrcr} & \multicolumn{3}{r}{12} \\ & \multicolumn{3}{r}{\times\ 14} \\ \hline = & \begin{array}{r} 12 \\ \times\ 10 \\ \hline \end{array} & + & \begin{array}{r} 12 \\ \times\ 4 \\ \hline \end{array} \\ = & 120 & + & 48 \\ = & 168 & & \end{array}$$

Step 5: **Experiences at the Abstract Level**

As we have said earlier that the concrete material should alway be used according to the principles of free activity, in which the child is encouraged and helped to discover mathematics. The same principle should guide us in the question of written and abstract work. If children have difficulty in doing mathematics using abstract notation we must conclude that their introduction to it is has been "dogmatic and inflexible." (Goutsard, 1964) Mathematics signs should not be introduced as absolutes and in isolation, they should emerge as a natural extension of the preceding activities. They should come in at the opportune moment, when they are helpful to understand a phenomenon under discussion and as an aid to make the study easier either in understanding or representation. After the children have had enough experiences at the concrete and recording level, they should be given several problems at the abstract level and experimentations and vice-versa.

L. Implications

Some educators, citing Piagetian developmental theory, abandon the use of manipulatives after the students reach the age of twelve. Bishop (1980) states that one can "see the evidence of an unfavorable influence in the secondary schools where teachers make the assumption that all students are at the formal operational stage of development. This is especially true for students who have visual perceptual deficits. Bruner concurns by stating

> It is true that the usual course of intellectual development moves from enactive through iconic to symbolic representation of the world, it is likely that an optimum sequence will progress in the same direction. When the learner has a well developed system, it may be possible to bypass the first two stages. But one does so with the risk that the learner may not possess the imagery to fall back on when his symbolic transformations fail to achieve a goal in problem solving (Bruner 1966 p. 54).

Counting and measuring depend on the ability to conceive of number as an abstract entity. Without this ability and concept, we would have to use a different word to express different collections containing the same number of objects with different words. This development of the understanding of number as an abstract concept is a process that reflects the ability of the child to interalize a large number of concrete experiences and then unify them. This is a cultural development. In our day to day living, we help the child to seek it out. The child sees it again and again and reaches the level of abstraction. Every normal child has this capability. What is needed is that the adults around him provide him these repeated experiences without being impatient and hurried. Due to lack of this abstraction ability, too many students rely on memorization.

Language is the finest example of abstraction. It is a means of expressing the conceptualized ideas in abstract and symbolic form. It is debatable whether language is the basis of abstractions or abstraction in thought is responsible for the production of language. There is evidence to support both points of view. We think it is the interaction of the two. Concrete experiences result in abstraction and language then soldifies the ideas and helps convey it to others. It is, therefore, important that the teacher regularly exposes children to the oral use of mathematical terms and at the same time asks them to verbalize their concrete experiences. The child who is only asked to give monosyllabic replies has difficulty gaining mastery of the mathematical language. He should be asked to give complete sentences and complete thought units as the answers to the questions. A child who is only asked to write just the answer will never master the language of mathematics. He must be encouraged to form complete expressions for himself instead of receiving them ready-made from the teacher. To illustrate this point let us consider the following word problem:

> The Black Caves are 442 miles from John's house. John's family is going to visit the Black Caves. On the first day they covered 329 miles. How far away are they from the Black Caves?

The child is asked to solve this problem. He finds the right answer 113 miles by subtracting 329 from 442. He writes the answer: 113 miles. At this point the teacher should have asked him either orally or in written form to give the full answer, namely: They are 113 miles away from Black Caves. This kind of complete sentence answers should be asked every day. Whether it is solving word problems or teaching forming word problems children should write everyday — for only by practice can one become an expert in the mathematics language. And the development of an idea at abstract level is demonstrated at the mathematics language level.

Once a child has mastered the concept under consideration, the teacher should use the same steps for the next concept.

These steps can be graphically represented as follows:

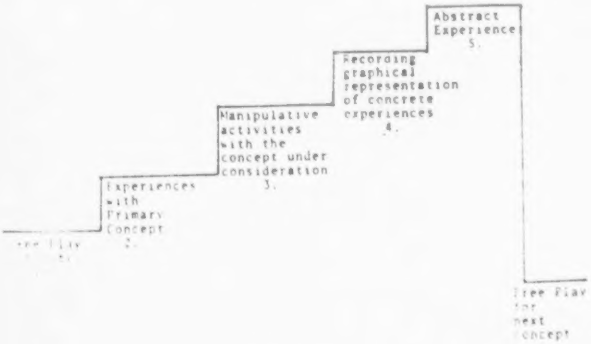


FIGURE 5

There is no single route to abstraction. The rate at which children pass through from concrete to abstract differs from child to child. It is a function of many factors: Physiological and anatomical maturation, physical experiences, and social interaction.

This model of concrete-to-abstract continuum is not only applicable to teaching, but can be applied to diagnostic evaluation of children's mathematical learning level.

M. Examples of Concrete Instructional Materials

Stern Blocks

This apparatus consists of a set of wooden blocks from one cube unit to ten cube units in length, with the unit length marked on each side. There is also a hundreds unit in the form of a slab and a 10 x 10 x 10 thousands cube. The blocks of the first decade are supplemented with various devices such as a "counting board" (for associating each block with its symbol), 'number cases' (from 1 x 10 x 10 units capacity for illustrating the synthesis and decomposition of numbers up to ten), a 'number track' (enabling linear combinations of blocks to a total of 100 units) and a 'dual board' (for illustrating place value as far as tens and units). Essentially, the material is used to model the addition, subtraction, multiplication and division of numbers up to 100, or using the hundreds and thousands blocks, and accordingly the material is primarily used in developing arithmetic concepts and procedures. Stern bases the theory of her system on gestalt principles and particularly on the views of Wertheimer. She claims that by carrying out arithmetical operations on carefully devised material, the structure of these operations will be more evident to the child than if the same operation are carried out symbolically. She also stresses that the child should make his own discoveries and not merely follow what other children to the teachers are doing.

Cuisenaire Rods

The Cuisenaire rods are similar to the Stern's blocks in that they range in size from a cube to ten times the cube's length, but they are smaller, being only 1 cm. Sections, are unsegmented, and are colored according to 'color families': one is in white, twos (2, 4, 8) form a red-brown family, threes (3, 6, 9) form a green-blue family, fives (5 and 10) form a yellow-orange family, while seven is black. Indentification of rods is not by counting, as it may be with segmentations in the Stern, but by color. The first association is thus between color and size and all initial manipulations are originally expressed in a color symbolism, e.g., b(rown) + r(ed) = o(range). This is done because it is believed that the child's attention is thus drawn to the relationship itself rather than to the difficulties inherent in learning a less obvious symbolism, i.e. 8 + 2 = 10. Later, of course, the conventional associations are made. Color is also to facilitate the understanding of multiplication, e.g. eight reds is more obvious perceptually than eight two-blocks. The lack of segmentation is thus believed to lead to a ratio concept of size preserving the cardinal unity of the class size rather than a lower order concept formed by breaking each emphasized by Stern: although her blocks are segmented, it is still easier to measure lengths against each other than to reduce each to ones. Fractions are dealt with very early in the Cuisenaire system, since the lack of segmentation permits any rod to become the unit. Again this is possible where the blocks are segmented, but the step may be easier to understand where there is clearly nothing to prevent a red from being a 'one'. The theory behind the Cuisenaire rods is basically a version of Piaget's theory. The child's actions with the rods are said to become internalized through repetition and thus aspire to the status of logical operations. All arithmetic operations and many of the simple algebraic concepts can be modelled by this material. We have found this material to be the most effective for the development of basic arithmetic facts.

Unifix Cubes

The unifix blocks do not in fact constitute a systematic method of teaching, although a manual has been prepared showing how certain relationships may be depicted. The material consists of interlocking plastic cubes that can be formed into structures similar to Stern blocks, and the use of the material resembles that appropriate to the Stern. All arithmetic facts can be modelled by this material, but we have found this to be the most effective material for the development of the concept of place value with the whole and decimal numbers.

Flavell Materials

This is primarily an extension of the Cuisenaire rods but involves more specific suggestions as to the means by which concepts may be applied to problem situations. Flavell also incorporates additional apparatus such as abacus. The theory behind this method is primarily Piagetian, as in the Cuisenaire itself, but more stress is laid on the importance of variety of experience leading to abstraction than is the case in the orthodox Cuisenaire approach.

Dienes's Blocks

The most important component of this apparatus is the Multibase Arithmetic Blocks (the MAB) which is intended for use to develop the concept of place value: this is followed up by the Algebraic Experience Materials (AEM). There are similar other components in the apparatus to develop other mathematical concepts. MAB consists of a series of blocks structured in geometrical progression to the third power in constant multiples of 3, 4, 5, 6, and 10, and it is used almost entirely to illustrate the operations of the four rules in connection with place value. The ten set in fact corresponds to the unit, ten, hundred and thousand blocks of the Stern. The purpose of using bases other than ten is that experiences leading to the abstraction of the concept of place value are provided, as opposed to illustrating one instance of the concept, i.e. the decimal system. In the card system accompanying the blocks, Dienes suggests that analogous or isomorphic structures should also be utilized in the form of exercises with ad hoc materials (boxes, counters, and pencil and paper games) and also in a less structured form with a kind of abacus where bases can be altered at will with a color notation. The structure of the material is also useful subsequently in that the linear, quadratic, and cubic functions thus modelled may provide some of the background for generalizing the basic concept of those of index and algorithms. Normally, however, the materials can be used with very young children but they are more useful and appropriate to a somewhat older age group than the majority of the other methods considered above. The theory underlying this method is rather complex but the basic priciple is that of abstraction through diverse experiences. Experiences are varied qualitatively, e.g. the logically analogous but perceptually different supplementary exercise with the boxes and abacus, and quantitatively, e.g. the insistence on working to different bases. Dienes also has ideas about both the logical and the psychological sequencing of learning and care is taken in the classroom presentations of the method to accomodate these features.

N. Summary

The instructional methods and materials, therefore, fall into three major groups: qualitative (continuous), quantitative (discontinuous), and mixed. Qualitative instructional materials are those in which the visual and spatial aspects are most important in deriving the mathematical structures and concepts. Mixed type of instructional materials are those which have both the features — qualitative and quantitative. Stern and Cuisenaire material fall in the qualitative (continuous) category, whereas, Dienes can be categorized as mixed. Unifix cubes, counting blocks, number line, invictabalance and abacus fall in the category of quantitative (discrete and discontinuous) materials.

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LEVELS OF KNOWING MATHEMATICS

MAHESH C. SHARMA

Every mathematics teacher wants his students to master mathematics concepts at the highest level possible. They want students to know those concepts with fluency and understanding. They want their students to produce the outcome of their mathematics learning, both concepts and operations, with ease and facility. But the teachers' perceptions and understanding about children's mastery of a mathematics concept and its manifestations vary. For example, some teachers may think that it is the mastery of facts and procedures that is important. These teachers may, therefore, emphasize the assessment of the mathematics mastery through standardized tests and examinations, other teachers may think that mastery means the mastery of applications of mathematical learning and that is what must be emphasized in teaching. Therefore, according to these teachers mastery is assessed through problem solving-exhibiting this through applications. Yet another group of mathematics teachers might put the importance of mathematics mastery on developing of interrelationships between ideas, procedures and the language of mathematics. The mastery of mathematics learning includes all of these and so many other aspects of mathematics and teachers' evaluation of children's mathematics learning will reflect that viewpoint. All of these must be considered as components for the required mastery of mathematics learning. Several methods must be employed to assess the mastery of learning in mathematics. But the teachers have emphasized different elements as components and type of mastery at different times in the history of mathematics education.

The mastery of mathematical concepts is a developmental process and is also a cumulative one. The complete mastery of any mathematical concept is achieved slowly and sequentially. For example, every mathematics concept consists of three components: linguistic, conceptual, and skill. Therefore, mastery in mathematics means mastery in all of these components. At the same time all mathematical concepts that a child learns are not mastered at the same level of proficiency and fluency. A child's mastery of mathematical concepts exists at different levels and in different forms. His mastery of every concept that he knows is not at the same level of fluency, efficiency and automatization; the levels of his mastery may vary along a continuum. A child's understanding of mathematical concepts, his ability in performing operations, his skill in applying mathematical facts and concepts and solving mathematical problems evolves slowly and in stages based on the types of learning experiences he has had. Of course, it also depends on the type of teaching provided to him and the level to which this teaching helped him to master the mathematical concepts. The child does not master any particular scientific and mathematical concept all at once; he reaches the full level of mastery gradually. In most cases it is developmental and it is also a hierarchical process.

The mastery of previous concepts effects the mastery of a future concept. At the same time the concepts that a child is learning and mastering are not merely a copy of what the teacher has presented. The concepts that have been presented to the child's conceptualization process and as a result of the student's active thought processes. Learning of a new concept, therefore, is the reorganization of old schemas (structures of knowledge) and the formation of new ones. Schemas that were present before the new learning experience have been reshaped to reflect the integration and assimilation of the new learning experiences and many new ones are shaped. Many old schemas are reinforced and solidified. New schemas also reflect the accumulated learning, knowledge, and skills. Concepts are not simply inserted into the student's store of knowledge; they are creatively remade by him. It is an active and dynamic process. The child is an active participator in the process. The child evaluates the new concepts and learning in the light of old concepts and knowledge. When there is an overlap and connection between the old and new concepts he accepts it and then it is assimilated.

Piaget's cognitive developmental theory helps us place the child's learning ability into an appropriate frame of reference. Children pass through several stages of cognitive development: sensory motor, preoperational reasoning, concrete operational reasoning and formal logical or abstract operational thinking. Before the child arrives at the abstract level of thinking his cognitive ability to handle information logically and to generate and create new knowledge, or ideas is quite limited. Before the child reaches the abstract operational stage he may have had diversity of information and experiences but he may not be able to interrelate them meaningfully and abstract and generalize ideas from these experiences. As a result they may not be able to develop new concepts and ideas from them. In the pre-abstract thinking children's understanding of concepts is limited to the specific situations and specific learning experiences, particularly related to those concepts. This means the mastery of that concept is also limited and is a function of those specific situations and learning experiences. Sometimes this learning is context specific or material specific.

Soviet educator and academician N.A. Manchinskaya's (1978) research on the psychology of mastering mathematical concepts has shown that the connections between a concept (or one of its features) and a child's past experiences, his emotional response and his active intellectual operations (levels of intellectual development) are the factors which, according to his investigations, determine the process the child uses in learning a mathematics concept and level of mastery achieved in that concept.

Piaget, in his important experimental and clinical work on the intellectual development of young children categorized the concept attainment and its mastery, according to the child's age, into three stages. The first stage is one of no understanding or having a vague idea of the concept and the third is one of complete understanding. The second stage is transitional and bridges the gap between the first and third stages.

The above mentioned stages proposed by Piaget and then supported by Bruner, in his restating them as **enactive**, **iconic** and **symbolic**, are too general and do not explain the phenonmenon of transistions from one experience to the other clearly. In our work with children and adults we find that the mastery of a concept exists at different levels and is a function of experiences with manipulations of concrete materials and concrete and thinking models being used by the learner in the process of learning mathematics. And these in turn reflect the teacher's choice and emphasis on teaching methodology, choice of instructional materials and models for conceptualization.

Although, once a child has completely mastered a concept, he has transcended the instructional material and methodology, but in the beginning stages of a learning experience, the child's mastery of the concept being taught is a function of these models and specific experiences. In other words the child's learning is a reflection of a teaching model and is experience specific. The child thinks the model of the concept and the concept being taught through the model are the same. The child has difficulty separating the concept and the model being used for learning that concept. In this stage of mastery the child may regress from the mastery level achieved to a lower level. If the concept is being tested through another model, mode, and experience the child may not be able to recognize the concept. Therefore, this level of mastery is quite limited. When the child transcends the models a higher level of mastery is achieved and that level of mastery is to a great extent quite permanent and nonregressive.

While observing chldren when they are engaged in mathematics learning we have found that they acquire the understanding of mathematical concepts at different levels of parital competence before the concept is completely internalized and mastered. We formulate these levels of understanding of mathematics concepts as:

1. **Intuitive** (enactive)
2. **Concrete** (iconic)
3. **Representation** (pictorial)
4. **Abstract**
5. **Application**
6. **Communication**

1. INTUITIVE/ENACTIVE LEVEL OF MASTERY

(Non-Verbal Learning Level)

During the sensory motor level of cognitive development the child learns at the motoric level. All his learning is through direct actions on the objects in the environment and his sensory experiences are the results of his direct actions. The knowing level, if we could at all call it 'knowing' is purely action (performace) oriented. He can perform many coordinated movements learned primarily through a stimulus-response arrangement; the learning that takes place can be characterized as more of trial and error manner of learning.

Adults, too, know many things at this sensory motoric level. The very young child has no language with which to describe his/her skills, and in many instances neither can the adult describe his motoric knowledge, even though theoretically he has the language to describe his motoric knowledge. For example, he cannot communicate to another human being how he walks. Many times when you ask an adult to describe an activity such a how one swims or rides a bicycle the only way one can describe such activities is through actual performance of that activity. Similarly when you ask a very young child what is 2 and 2 he will say I will show you and goes about showing that activity. Such knowing is termed enactive." (Underhill, 1972)

Apart from the individual motoric activities, the child also learns the relationship of different motor activities with each other. He begins to form associations between different activities. The intuitive state of learning is that state when the student becomes aware of a concept before a name has been assigned to the concept. Intuitive state is that state when the student is involved in the experience without a specific goal towards a concept; it is the preparation stage for specific learning. For young children it is the period of "free play and activity." "Intuitive implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's craft. ... It involves the enbodiment or concretization of an idea, not yet stated, in the form of some sort of operation or example. (Bruner, 1960)

The basic ideas that lie at the heart of all science and mathematics and the basic themes that give form to life and literature are as simple as they are powerful and initially they are all formed intuitively without apparent elaborate logical structure to back it. It is only when such basic ideas are put in formalized terms as equations and elaborate verbal concepts, that should have been first understood intuitively, that the child begins to have difficulty in learning a concept. When a concept is clearly seen at the intuitive level then the child is ready for its understanding at other higher levels. The difficulty in understanding is much more pronounced, particularly when the child did not have enough concrete experiences before it had a chance to try them out on his own.

During the pre-operational stage of Piaget, the child, through his motoric experiences, explores the environment and gains intuitive understanding of the processes and the underlying conceptual content. When questioned about a completed task or activity, a child's typical response is "I don't know how it is done, I just did it." This state of knowing is parallel to an adult's statement such as: "I have a feeling (idea) about how it's done but don't know how to show or explain it to you." This statement indicates and exhibits the state of "unstable equilibrium" when the learning and understanding is in the fluid state and more experiences are needed to make it meaningful and stable. With more and more specific experiences the idea will become clearer and will occupy a definite place in the conceptual frame work of the learner.

Thus, "the basic approach to all new subject matter should be intuitive," Kline (1973) believes. He also asserts **that "... there are specific and explicit aids to the function which enables it to function." Primarily, one makes sense of mathematics through sense, for as Aristotle first put it, "there is nothing in the intellect that was not first in the senses" — though Leibnitz added, "except the intellect itself."**

2. CONCRETE LEVEL OF MASTERY

In the natural development of the thought processes children first understand the concept at a concrete level-through the manipulation of the materials in their natural environment. At this level of understanding children think by first actually manipulating objects then forming mental images of concrete objects. When asked, "What is 2 and 3?", a young child counts two objects and then three objects and gives the answer. This concrete activity serves the basis for when a child pictures mentally the collection of two blocks and three blocks to make five blocks. The concrete experience was essential for the formation of mental imagery. It is not uncommon to get responses such as "I cannot tell you how it works, but I will show you the way it works." This means the concept is known to the person at the concrete level. When a child can give you the answer when actual objects are present but cannot give the answer to the same problem when you remove those materials from the child's visual field it shows that his mastery is limited. His mastery of the concept is limited to the concrete level of mastery. To reach the conclusion that he does not know because he cannot give the answer without the materials is as wrong as assuming that the child knows the concept completely because he can give the answer with the help of materials.

The reliance upon physical things to produce a feeling of certainty about the symbolic meaning of a problem is illustrated by a ninth grader who was somewhat doubtful about nine being the correct answer for 5 -(-4). She said, "Show me that it works in dollars and cents and I'll believe it." This demonstrates the concept of integers (an abstract concept) is a difficult one for this student as she is still functioning **at a concrete level not only in this but in many other earlier level concepts. And the only way she understands the abstract concepts, at this time, is through concrete materials. A child's competence or mastery at the concrete level is limited to exhibiting the phenomenon through concrete manipulation and demonstrating it through these materials. The child can identify the numbers by counting; and using counting cubes, rods and beads; subtract by physically removing objects from a given collection, etc. Visual memory and perceptual organization describe and dominate this type of knowing which is in Bruner's terminology known as the iconic form of knowing.**

Inhelder, who has worked closely with Piaget for many years, agrees with the idea that before children acquire mastery in a particular concept at an abstract level they do so first on a concrete level. She says, "...Basic notions in these fields (geometry, operations) are perfectly accessible to children of seven to ten years of age, provided that they are divorced from their abstract mathematical expression and studied through materials that the child can handle himself." (quoted in Bruner, 1963) Piaget, himself, in his several speeches, emphasized the importance of concrete materials. **He used to say, and I paraphrase, any concept can be** learned by a child if and only if the concept is presented through concrete manipulatives and activities and the child is functioning at the appropriate stage of cognitive development. Roughly twenty percent of students (both children and adults) referred to our center for mathematics remediation respond quite effectively and almost imediately (only in less than five sessions) as we present the mathematics concepts through concrete manipulative materials. Many of them will comment such as, "Oh! this is what it means." "Now I understand, now I can see it what my teacher was trying to show me through those numbers." The reaction is the same both from children and adults.

Concrete experiences and mastery of a concept at a concrete level build foundations upon which later, more sophisticated learning can and should occur. Children and adults learn by abstracting ideas from the experiences in the physical world. The physical "movements and operations are a preparation for conscious reasoning as they reproduce and prepare new manual operations of which thought is a continuation." (Piaget, 1928)

3. REPRESENTATION/PICTORIAL LEVEL OF MASTERY (Symbolic I)

Concrete experiences relating to a particular concept, intuitive understanding of that concept and its mastery at a manipulative level are not enough for complete understanding of the process or ideas involved in the concept. It is a natural need, if we want to communicate it to others or if we are looking for the validation of its mastery, to represent those ideas into some kind of symbolic form. When a child can manipulate 3 cubes and 2 cubes and finds that their sum is 5 cubes, he is working at a concrete level; his competence is being exhibited at the concrete representational level. But, when the child can demonstrate the experience through **a picture of 3 cubes + 2 cubes then he is showing that he** has the ability to represent his thoughts and understanding through representation; which is a higher level of understanding than all of the previous ones. Visual representation of concepts learned through concrete materials is a key step in mastering a mathematics concept at the abstract level.

The child's personal visual representation experiences have a definite effect on his ability to master concepts. The key idea here is that the pictorial models are being used as a means of abstractions. And as children can become material specific at the concrete level (he thinks the concept is the function of the model), it is possible that the child might take the same attitude about the visual models also. Therefore, instead of a single diagram or a single model, the teacher should use different diagrams to make the generality of the concept clear. For example, when the teacher always represents the right triangle in the upright position, many children think that a right triangle is always represented in that form and they are not able to identify the right triangle when it is embedded in other figures or is shown in nonstandard forms.

Morelli and Schwartz (1977) in their study on the effect of pictures and imagery on learning abstract concepts found that "a picture of the abstract-concrete word pair enhances its learning over the no picture condition... Pictures provide a qualitatively superior sensory code in constrast to their word labels... Presenting a particular picture to subjects in the learning of the abstract word-pairs may help by preventing image competition, thereby enabling subjects to use a particular sensory code." Although the above research was conducted in connection with the learning of words, particularly nouns, but the principle of using pictures-symbolic representation of concepts and ideas - is equally applicable to the learning of abstract mathematical concepts. The picture of the abstract stimulus will serve to make the abstract item more concrete, thereby enhancing the learning process over abstract ideas with no pictures. "Pictures directly arouse images which may then serve as mediators in learning and memory processes."

About the interaction between concrete and visual representation, Werner (1940) says, "The child's concepts always have a concrete content. Image and concept are an indivisible unity. The conceiving and describing of a thing are not distinctly separated activities." Furthermore, what teacher would dream of teaching geometry without drawing pictures of lines, triangles, and points on the chalkboard? The appeal to the physical world through pictures and drawings, enables the student to take a first step in his study of mathematical systems.

Teachers of children up to age 9 must have observed the phenomenon that many of these young children do not listen or read the directions related to mathematics problems. They look at the pictures in the textbook or observe their neighbor's clues for the operation under consideration. Pictures tell them what to do. If the birds are flying away from the paper, it must be a subtraction problem. If they are coming towards the page it must be an addition problem. If there is a large circle around a number of objects and then there is a small circle inside the larger circle with fewer objects than the large circle, then it must involve subtraction. This shows that these children are functioning at a pictorial or representation level of mastery. As another example, many students at the ninth and tenth grade level can furnish the proof for a particular theorem when the theorem is presented with an accompanying diagram. But many from this group will have difficulty furnishing the proof if the accompanying diagram is absent.

When a student can answer the problem with the help of pictures, diagrams and context, and if the pictures, diagrams and the context to the problem are absent and the student can not answer the problem that shows that the student is functioning at the pictorial or representation level of knowing.

4. ABSTRACT LEVEL OF MASTERY (Symbolic II)

Continuing with the previous example, if the child is representing his concrete experiences through representations such as:

3 apples + 2 apples = 5 apples
3 cubes + 2 cubes = 5 cubes
3 steps + 2 steps = 5 steps, etc.

and discovers the abstraction

3 of something + 2 of the same thing = 5 of the same thing

and derives the equation

$$3 + 2 = 5$$

and finally understands that all the above are the special cases of

$$3\square + 2\square = 5\square$$
$$3x + 2x = 5x$$

then the child shows that his understanding of the concept has reached the abstract level.

In other words, the child is able to perform the operation of addition at the abstract, symbolic, or formal operational level. Operations at the formal or symbolic level do not require immediate concrete referrents, but the previous concrete experiences help the child to perform mental manipulations on abstract symbols that occur in highly organized and symbolized forms of mathematics. The more experiences that the child has at the concrete and pictorial level, the better frame of reference he is going to have in visualization for his abstract operations. **Concepts do not originate with the abstract definitions, but they do not end with them either. But, this is the level where most of the teaching takes place. Most teachers devote most of their time at this level of teaching. Most of the materials in textbooks is at this level. Most of the examinations and tests take place at this level.**

During instruction one frequently encounters cases where both a concept and an object (or its visual representation) which is to be understood by means of the given concept are introduced simultaneously. We find a simultaneous movement in opposite directions, from concrete to the abstract, and from abstract to concrete. That is, abstraction and perception occur in parallel. This is very clearly reflected in the content of the concepts during the initial stages of their mastery.

The very point of an abstract formulation is that it unifies and reveals common properties in concrete and familiar branches of mathematics. Therefore, contrary to the common traditional classroom practice, abstraction is not the first stage but close to the last stage in mathematical conceptualization and development. It gives insight, but only into concrete structure already well learned. It unifies, but only what one already knows. Without much practice and "without much previous knowledge of concrete cases," says Morris Kline, in his famous book, **Why Johnny Can't Add**, "abstract concepts remain empty, arbitrary children of mathematical fantasy. To confront youngsters with abstractions that lie above their level of maturity is to create bewilderment and revulsion rather than increased knowledge."

The presentation of a concept has to reflect the level of the learner. The nature of the concept at an elementary level should be in concrete form. The highly abstract concepts in their full complexity can not be explained at an elementary level. Similarly, an abstract concept at the secondary level with some concrete and visual models can transcend to an abstract level if students are able to use effective use of that limited experience. But, if the students do not have the concrete model to conceptualize it, they will just memorize it. Before the teacher presents it at the abstract level, she should make sure it is understood at the concrete level or the student understands it by identifying it by the help of some patterns. Even on the secondary level, for example, the concept of algebra of matrices shows the implications of teaching at an abstract level. Students are taught how to add, subtract, multiply and divide matrices as well as many other operations. However, matrices are themselves abstracted or abbreviated forms of what are called transformations. **Many students, even at the twelfth-grade level, have no background in the significance and uses of transformations. Hence they learn merely to manipulate matrices to no purpose, and they cannot have any feeling for or make** sense of what is accomplished because they do not know the context and concrete cases in which matrices are useful. The manipulation of matrices, then, is a series of mechanical tasks - which do not remain with the students for long.

Professor Kline further emphasizes the point that putting too much importance on abstraction too early is harmful. He adds, "Psychologically the teaching of abstraction first is all wrong. Indeed, a thorough understanding of the concrete preceeds the abstract. Abstract concepts are meaningless unless one has many and diverse concrete interpretations well in mind. Premature abstractions fall on deaf ears... Abstractions must grow on people. They cannot be handed down **ex cathedra**. As people's experiences with different varieties of functional relationships and their peculiarities increase, they will gradually see the wealth of situations which have a basic idea in common; at this stage they will be enlightened if this common idea is pointed out. They will also appreciate the various conditions or qualifications that must be incorporated in a general definition."

Professor Kline and others strongly advocate the understanding of the concepts first at the intuitive level. This is because the intuitive approach also includes the appeal to the heuristic arguments. Through experiences with actual objects a child can learn that $3 + 4 = 4 + 3$. The generalization that $a + b = b + a$ is heuristic. Another aid with is also an integral part of the intuitive approach is reasoning by analogy - a skill of great advantage. The intuition can be inculcated and sharpened through physical experiments and the concrete example of a concept can be built based on an intuitive feeling of the idea. All of the above devices, pictures, heuristics arguments, induction, reasoning by analogy, and physical arguments are appeals to the intuition. Of course, the intuition about what to expect in human behavior improves with experiences and so does the mathematical intuition.

Clearly the intuitive/discovery approach can lead to error also, but commiting error and learning to check one's results are a part of the learning process. "Truth," Francis Bacon noted, "emerges more readily from error than from confusion." If the fear of errors is to be a deterrent a child would never learn to walk; and a student who makes no mistakes makes nothing else either. The intuitive state of knowing therefore is a state of exploration many times without any specific aim and conceptual outline.

5. MASTERY OF A CONCEPT AT THE APPLICATIONS LEVEL

The process of mastering and applying concepts are inseparably linked with each other. In our opinion, if a pupil has really mastered a concept, he should be able to use it. Conversely, the inability to use a concept clearly testifies that the concept has not been mastered completely or that inessential features have crept into the content of the concept as understood by the pupil. By using a concept, especially in diversified cases and conditions, we understand it better; we attain a more fundamental and correct mastery of it.

Applications, according to Luriya, of a concept can be considered in two ways: as a means of fuller elucidation of the concept, and as criterion for judging whether the student has really mastered the concept.

Concepts also arise in a variety of systems in the real world. Once again it is an interactive process. Applications of mathematical concepts both enhance and test the mastery of a particular concept. The best example is the process of estimation: it tests the concept being estimated and it also helps to learn that concept. There is indeed, a close connection between acquiring concepts and making estimations. Estimation is also a form of application of a concept. Every concept can be divided into constituent judgement (estimations), and each judgement into a concept (Usinskii).

Application of a concept takes place at four levels: Internal applications, external applications, creating new ideas and concepts and problem solving.

Internal application: Let us assume that the child knows 4×5, 3×1, 3×5, and 4×1 as multiplication facts, the concept of place value, and the estimation and now he is introduced to the multiplication algorithm with an example such as to find the produce of 43×15. If he is able to compute 43×15, then we will say that he can apply single digit multiplication facts to two digit by two digit number multiplication.

External application: This means the application of a skill from one content area to another. For example, when the child can apply the knowledge of ratio and proportion to, say, an analysis of a chemical reaction (titration). In titration two solutions (one acid solution reacts with alkaline solution to neutralize each other) react, the objective is to find the strength of the unknown solution with the help of information about the known given solution.

Creation of a new concept or information: When a child on the basis of known information creates new information for himself by combining known concepts and ideas. For example, he knows the concept that an isosceles triangle has two equal sides, but on further exploration finds that the opposite angles (to the equal sides) are also equal, or the two sets which are in one-to-one correspondence contain the same number of elements. These examples illustrate the point that the child needs to apply his knowledge to discover and create new concepts.

Problem Solving: When a child can solve a word problem involving the concept(s) which he has mastered at the previous levels we say he is applying concepts or skills. Experience tells us that when children have actually mastered the concept, they can apply it to different and new situations. However, one also encounters those students who can solve only the exact kind of problem that they did in class with the teacher's help and at the slightest modification of the problem they are helpless. It seems that more often than not, rigid, formal instruction makes the students unable to use their knowledge in practice. When children are shown different variations of a problem and are made aware of the essential features of a concept, then they are able to apply a concept easily. In essence, this means that the intellectual operation of subordinating the phenomenon to the concept plays a great role utilizing knowledge. In performing this operation, one must separate the essential from the nonessential and contrast them to each other. E.N. Mellar's (quoted in Menchinskaya) investigations have shown that in teaching students with weaker preparation, one must especially point out the essential properties of the phenomena covered by the concept being studied, and what their possible variations are. Word problems, real life applications, projects, interdisciplinary studies and projects provide children opportunities of not only applying the concepts in mathematics but also raw materials, concrete models, analogies that will serve them as examples for understanding future abstract mathematical concepts. Some mathematics is discovered from out of nowhere, from the mind of the mathematician, but most mathematics is discovered either as a result of solving problems posed to mathematicians by others or by themselves. Students should be exposed to this dual nature of mathematical applications: applying mathematics to mathematics and then applying mathematics to real world.

6. COMMUNICATION LEVEL OF MASTERY

The child who has mastered the concept at the intuitive or enactive level, when asked "how did he do a certain thing?" will say, "I don't know." The child who has mastered the concept at the concrete level will answer the same question as "I can show you how I did it." And the child who has completely mastered the concept will be able to communicate the concept (explain the procedures and ideas).

The ability to calculate does not necessarily require the ability to describe the calculations. It is possible to conceive of the case of a child who can perform short division calculations without being able to express the process. Piaget has found that young children often cannot introspect on their mental operations. If you ask them how they solved a given problem, they often cannot tell you. Real mastery is the outcome of this introspection through reflective thinking and then through its communication. Real understanding of a concept implies more than mere calculations. "It is possible to speak about a 'concept' only when knowledge reaches verbalization" (de Quiro's and Schrager, 1978). Another way this level of mastery is reached is when the child has the opportunity to explain to some body what he has learned. For example, in peer tutoring situations a student has that opportunity. The non-homogeneous groupings provide this opportunity more often than when children are grouped by homogeneous criterion. For certain period of school time children should have the opportunity to interact with other children of different levels of mastery and cognitive levels in order for them to reach the higher level of mastery in acquiring mathematical concepts. In peer tutoring, both children (the child with lower ability and lower cognition and the child with higher ability and higher cognition) have the opportunity and conditions conducive to learn and we have seen that both do learn. It is easy to understand why the child with lower cognition would have the opportunity to learn, because he has the advantage of interacting with a child of higher cognition. He has the opportunity of watching the strategies that the higher level cognition child uses and as a result of this he begins to see the use of new strategies. Eventually he incorporates these new strategies in his repertoire of strategies. But, "what is in it for a child who is of higher ability and is functioning at a higher level of cognition?" I am asked constantly by parents and teachers whenever I advocate for non-homogenous grouping, peer teaching, and group projects. My answer is: the child with higher cognition has the opportunity of communicating the concepts to another person; he has to think of models of explaining the concept he knows, that requires thinking; He reflects on the concept in order to explain it to somebody else; and this activity (of teaching some other person) solidifies his understanding of that concept. The child with such experiences is able to retain the mastery of the concept and faces very little regression. Almost every teacher will agree with me that once you have taught a topic you have mastered it. Many parents of able children object to their children being paired with children of lower abilities in peer teaching and joint projects. Teachers should explain the importance and benefits of this type of learning experience.

Another way the communications level of mastery can be achieved is by asking a lot of oral questions during teaching so the child has the opportunity to communicate what he knows at a higher level of understanding and get immediate feedback for the answers.

The successive levels of masteries do not always necessarily appear in the same sequence for each mathematical concept and then disappear; rather, they are functions of a child's experiences, his cognitive development, and the teacher's concern for mastery in learning. In some cases they may appear almost simultaneously in the acquisition of different concepts. For example, a child might have mastered the concept of addition at the communication level, subtraction at the abstract level, multiplication at the concrete level and Euclidean geometry concepts at the intuitive or enactive level.

In essence, as a child passes through different stages of cognitive development, he develops new tools and strategies for dealing with his environment and his mastery levels become more and more sophisticated. Cumulatively, these skills (or levels of skills) enable him to transit from one stage to other smoothly and with appropriate abilities and thereby able to expand his knowledge. (Underhill, 1972)

IMPLICATIONS FOR LEARNING AND INSTRUCTION

New learning generally appears to be developed sequentially through these levels of skills. Many teachers teach a concept at a specific level. For example, many elementary school teachers introduce mathematical concepts at the concrete level but do not provide adequate strategies to record these experiences at the pictorial and symbolic levels right after the experience and as a result the child's mastery level remains at the concrete level. On the other hand some teachers begin their teaching at the abstract level and some of the children may not have had the corresponding concrete experiences or even if they had the experiences before, they may not be able to recall the appropriate concrete models to understand that concept properly. Both of these are inadequate strategies. If the concrete experiences are not recorded into pictorial and symbolic levels, then the child may not see the abstractions and generalizations of that experience. The experience then remains at the concrete level. Many well intentioned teachers report to me that they use a lot of concrete materials and experiences but their children still have difficulty and continue to use the concrete materials even when they may not be necessary. The reason for this difficulty in most cases is two fold: first, the inappropriate material; and second, the lack of immediate recording of the concrete experience. The other extreme of this situation is when the teacher does not provide any or very limited appropriate concrete experiences. In such a situation, the student has to develop the concrete model of

the concept himself in his mind and this model may or may not be appropriate and efficient. For example, many children use their fingers as a self-devised concrete model for calculations. This is an inefficient and inappropriate model.

In working with children, teachers should use the Five Step Spiral approach: free play, concrete activity, symbolic representation of the concrete activity, abstraction and generalization of concrete activity, presentation of idea in the abstract form, and free play for the next concept—in the introduction of mathematical concepts. (Sharma, 1978)

The understanding that children progess through several levels and degrees of knowing a mathematical concept, skills or knowledge has two major implications. The **first** relates to teaching. Teachers' understanding that mathematics concepts should be presented at different levels beginning from intuitive level of presentation to abstract level. The **second** relates to grouping of children in a classroom. The grouping of students should take place according to their cognitive levels of development and level of mastery they exhibit in knowing mathematics. A problem of continuing concern in the teaching of mathematics is that of adjusting instruction to the varied abilities of pupils in the average classroom. We are all aware of the fact that the children with varying abilities are found in every classroom. One of the reasons for this variance is that they are at different levels of mastery in knowing mathematics. Therefore, one way of dealing with the problem is to group children according to their levels of knowing along with considerations for varying the pace and depth. Teaching by levels of learning is made easier by this type of grouping, but this grouping is unlike the conventional grouping of "fast" and "slow" learners. Grouping, in this form, is varied and flexible. Pupils will group themselves differently for different topics. An obvious conclusion of this approach, as Bruner (1960) says, should be that in our curriculum planning we should "opt for depth and continuity... rather than coverage". We should "give the child a sense of his own growth and his own capacity to lead ahead in mastery."

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DIVISION: HOW TO TEACH IT?

(MAHESH C. SHARMA)

The concept of division is a difficult concept for most children. The division algorithm is even more difficult for many children. The concept of division is a "secondary concept" [A secondary concept is a concept that is dependent on several primary concepts. Primary concepts are those concepts that are independent of other concepts and involve only one idea in them.] The concept and the skill of division involves: estimation, multiplication, subtraction, comparison, remainder, sequencing, etc. Difficulty in any one of these primary concepts may create difficulty in division also. Therefore, before the concept of division is introduced it is important that all the related primary concepts and preskills are reviewed so as to ascertain that the child has mastered these necessary preskills and the related primary concepts. In other words, just before introducing the concept of division, the teacher should review these concepts by devoting a few minutes to each one of them to make sure that the children are ready for the concept of division.

One should remember the important principle about learning that when we introduce a new concept to a child, he will be much more receptive to the new concept if the new concept has a high overlap with some old concepts, knowledge or skill that the child has already mastered. Therefore, every new concept must be introduced as an extension of something that the children already know. In this case, the division concept is related to multiplication, subtraction, and grouping, therefore it is important that it should be connected with those earlier concepts.

According to learning theorists such as Piaget and Bruner, understanding and mastery of every new concept passes through the following stages: evaluation, accommodation, assimilation, and incubation. Evaluation means that the child appraises the new concept in the light of what he already knows. If there is a high overlap between the new concept being learned and old concepts the child already knows, then the evaluation results in acceptance of that concept and then the child is subconsciously willing to accomodate the new concept in his old schemas of learning. This accomodation results in assimilation of the new concept with other concepts and then the incubation of that idea takes place. If the new concept has no or very limited relationship with previous concepts, the evaluation results in negative. Naturally, then the accomodation and assimilation do not take place and, therefore, no or limited learning is possible. All of these processes, therefore, result in an optimal state if we connect the new concept with something that the student already possesses. For optimal learning, the evaluation must result in positive. Thus, the concept of division will be better understood if we connect it to the primary concepts and prerequisite skills related to it, particularly when it is connected with concepts such as estimation, addition, subtraction, and multiplication.

A: Multiplication vs Division

To understand the concept of division well, a student should have conceptualized the concept of multiplication well and should know multiplication facts in almost automatized form. The concept of division is the inverse of multiplication. And since multiplication can be seen as repeated addition, therefore, division can be seen as repeated subtraction or repeated grouping. Understanding of the concept of division is intricately related to the mastery of the concept of multiplication. The student should have a very good model for the concept.

The usual definition of multiplication given in the lower elementary school grades is in terms of repeated addition only. For example, when we want to find 9 x 5, we count 9 sets of five or five sets of nine. This definition is highly sequential and counting based. With this definition, the only skill needed to find the answer and to conceptualize the idea of multiplication is sequential counting. It is a limited definition. It works only for whole numbers but does not prepare the child to conceptualize the multiplication of two fractions and other types of numbers. Children, therefore, find that their understanding of multiplication of whole numbers to be of little help in conceptualizing the multiplication of fractions and decimals. Because of this lack of preparations in the total understanding of the definition of multiplication they are not well prepared in understanding the concept of division. The concept of repeated addition easily applies to the definition of multiplication, the concept of repeated subtraction does not apply to division that easily. The skill of counting which has been a major skill in developing concepts up to this point does not come to the help of the child in understanding division. And the sequential counting skill does not apply to fractions either, therefore, children have difficulty in understanding the concept of multiplication and division of fractions also. For example, if we want to find the answer for 1/2 x 1/3. The definition using sequential counting does not work. How do we count 1/2, 1/3 times? Or how do we count 1/3, 1/2 times. We have to resort to a new model, or a new definition which is quite different than the model that the teacher in the second and third grades used for multiplication of whole numbers. And a little later, when we have to conceptualize a x b, the definition that we used for fractions or decimals is again inadequate. What is lacking here is a definition/model that can help a student to conceptualize multiplication whether he is dealing with whole numbers, fractions or algebraic expressions. Unless a student sees a good (all encompassing) model he will have difficulty in conceptualizing the idea of multiplication and little later the concept of division. [For a more detailed discussion of the topic of multiplication and how to teach it see **Math Notebook**, volume **1**, nos. 9 and 10.]

B. Concept of Division

The other major problem in understanding the concept of division is the lack of a good concrete model in order to visualize and understand the concept of division. The teaching of division requires both quantitative and qualitative models for the understanding of the concept of division. The popular models of teaching both multiplication and division are quantitive and are dependent on the concept of repeated addition or on sequential counting. The quantitative models of teaching are based on sequential counting such as the number line. The qualitative models of teaching are based on visual/spatial representations using instructional materials such as: Cuisenaire rods.

Popular Model:

To illustrate 12 ÷ 4, generally the teacher uses descrete materials. She demonstrates it by showing 12 objects

* * * * * * * * * * * *

and then makes groups of 4 from the above collection as shown below:

* * * *   * * * *   * * * *

And then infers that the answer is 3. Thus, 12 ÷ 4 = 3. Some teachers add the corresponding language model to it. They ask:

How many groups of 4 can we make from 12?

Here the key phrase is "how many groups?" We should add a third element to it: the qualititive element. Find, 16 ÷ 2.
— — — — — — — — — — — — — — — — divided by — —

To do so, the teacher takes a 16 unit long paper strip or an orange rod and a dark green rod (Cuisenaire Rods) place them next to each other length wise. And then she takes a 2 unit long paper strip or a red Cuisenaire rod. Then she actually demonstrates how many 2 unit long paper strips or red Cuisenaire rod can fit the 16 unit long paper strip or the lengths of 1 orange rod and a dark green rod. Or how many red cuisenaire rods can fit the combination of an orange and a dark green Cuisenaire rods. Through this model children can clearly see the concept of division qualitatively. In other words, the key element here is to see that we are making groups of 2's from 16 or more simply we are subtracting 2 from 16 repeatedly (as in the quantitative models).

After this demonstration, the teacher should ask the children to represent concrete models for the given division problems. For example, ask children to show the following division problems using Cuisenaire rods.

i) 18 ÷ 3, ii) 24 ÷ 6, iii) 50 ÷ 10, etc.

C: Division Algorithm

Although, division is an abstract concept, still it can be shown concretely. But the algorithms of division is even more of an abstract process involving several sequential steps and operations. On the continuum of the development of mathematical algorithms, in the historical context, it is a very recent phenomenon. Although the concept of division was known for antiquity, yet the long division algorithm, that we are most familiar with, is not more than 500 years old. It is many years of work by the top mathematicians of those times to get this elegant, though difficult algorithm. It is very compact, abstract, sequentially oriented, and complex. Only mathematicians can really appreciate the beauty of this procedure. Therefore, this algorithm requires a sophisticated level of understanding on the part of children. As it is clearly evident by just observing it that several preskills and primary concepts are involved.

(1) Given a division problem writing it in terms of its multiplication equivalent:

$$\begin{tikzcd}
\overset{\text{(i)}}{23\overline{)690}} \arrow[r] & \overset{\text{(ii)}}{690 \div 23 =} \arrow[d] \\
\underset{\text{(iv)}}{\square \times 23 = 690} & \underset{\text{(iii)}}{23 \times \square = 690} \arrow[l]
\end{tikzcd}$$

Ask children to read the given problem (i) and ask them to convert it into the division from (ii). And then from (ii) to (iii) and (iv). Now ask them to read the problem in multi-language expressions. For example, six hundred ninety divided by twenty three equals what; or 23 divides into 690 equals what; or how many groups of 23 are there in 690. And finally ask them to convert the division form (ii) into the multiplication form (iii) and (iv). Ask children to read the multiplication equation: 23 times what will give us 690; or how many 23's will make 690, etc. Now ask them to estimate the factor in (iii) or in (iv) and translate back this estimate to be the estimated quotient.

(2) **Estimation:**

Given a division problem ask the children to write its equivalent in the form of estimated division problem:

Division problem ◆ estimated problem
12)915 ◆ 10)900
915 ÷ 12 = ☐ ◆ 900 ÷ 10 = ☐

The estimated quotient is 90. After this ask them to give you the estimated answer. Several dozen problems of this type must be done before the whole algorithm is introduced.

(3) **Importance of the concept of remainder:**

The idea of comparison after each division step in the division algorithm is crucial. The students should clearly understand the reasons for the remainder to be smaller than the divisor. This point is not emphasized enough. The introduction to the concept should begin at a concrete level by using Cuisenaire rods or other similar material and then transferred to pictorial level as follows:

1) Divide 7 by 6.

— — — — — — — —

— — — — — — —

We make a group of 6 from 7 and as we can see there is one remains after making a group of 6. Place a green Cuisenaire rod (6 units) next to a black Cuisenaire rod (7 units) and then one shows the remainder in the black rod.

2) Divide 10 by 3.

— — — — — — — — — —

— — — — — —

Which one of the above examples shows the correct remainder. Why the process shown in the second case does not show the correct remainder? These type of examples must be discussed at great length to make sure that children understand the concept of remainder clearly.

(4) **Covering number strategy**

An important thing to remember at this point is that the division algorithm is a procedure and therefore children need supervised practice in this procedure. After enough practice at estimation, we can shorten the steps in (2) by the following strategy:

23)590

Ask the following questions:

T: Does 23 divide into 5 (we are pretending that it is 5, actually this 5 represents 500)?
S: No.
T: Put a 0 over 5 above the dividing line

```
  0
23)590
```

T: Does 23 divide into 59?
S: Yes.
T: Cover 3 and cover 9 with your fingers, and tell me how many times 2 divides into 5?
S: 2 times
T: Put 2 over 9 above the dividing line.

```
  02
23)590
```

T: Now multiply 23 by 2.
S: 46.
T: Is 46 close enough to 59.
S: Yes.
T: Is 23 × 3 bigger than 59?
S: Yes.
T: 2 is our estimated quotient at this stage, which actually represents 20. But we will treat it as if it is 2. Put 46 below 59 and subtract 46 from 59. What is the result?
S: 13.
T: What is this number called?
S: Remainder
T: Is the remainder 13 smaller than 23 (the divisor)?
S: Yes.
T: Good. What would have happened if the remainder was bigger than 23?
S: We would have tried 3 in place of 2 and would have multiplied 23 by 3.
T: Very good. What do we do now?
S: Bring down 0 next to 13. (If the students are not able to give the answers as we articulate them then you provide the answer with some discovery questions.)
T: What do you have?
S: 130.

```
  02
23)590
  -46
   130
```

T: Cover 3 in 23 and 0 in 130. How many times does 2 divide into 13?
S: 6 times.
T: Multiply 23 by 6. What to you get?
S: 138.
T: Is it bigger than, equal to, or smaller than 130?
S: bigger than.
T: Is our estimate 6 right?
S: No.
T: What shall we do?
S: Try 5.
T: Put 5 over 0 above the dividing line. And multiply 23 by 5. What do you get?
S: 115.
T: Is 115 close enough to 130?
S: Yes.
T: Put 115 below 130 and subtract. What do you get?
S: 15.
T: Is the remainder smaller than, bigger than, or equal to 23?
S: Smaller.
T: Is there any numeral left in 590 that has not been used?
S: No.
T: Are we done?
S: Yes.
T: Read the problem and the answer.
S: 23 divides into 590, 25 times and 15 is the remainder.
T: Read the problem and the answer in another way?
S: 590 divided by 23 we get 25 and 15 is the remainder.
T: Write this as a division problem using ÷ sign for division.
S: 590 ÷ 23 = 25 and remainder 15.

T: Write this as a multiplication equation.
S: $590 = 23 \times 25 + 15$.
T: Write the following division problem as a multiplication problem

a) $\begin{array}{r} 7\text{ R }1 \\ 9\overline{)64} \end{array}$ $\qquad$ $\begin{array}{r} 21\text{ R }7 \\ 21\overline{)448} \end{array}$

S: (a) $64 = 9 \times 7 + 1$ b) $448 = 21 \times 21 + 7$
(Ask about a dozen such questions)
T: Write the following division problem into a multiplication problem

a) $\begin{array}{r} c\text{ R }d \\ a\overline{)b} \end{array}$ $\qquad$ b) $\begin{array}{r} \$\text{ R }@ \\ \#\overline{)**} \end{array}$

S:
a) $b = a \times c + d$ $\qquad$ b) $** = \# \times \$ + @$
T: Write the following multiplication problem as a division problem:
$424 = 21 \times 20 + 4$
S: $\begin{array}{r} 020\text{ remainder }4 \\ 21\overline{)424} \end{array}$

Teachers should provide several exercises like the above to help the children see the idea of division. The above question-answer process can be completed in more than one session. The objective is to reach the level of conceptualization indicated above.

(5) Discussion of the estimated answer and the actual answer.
Before children actually calculate the actual quotient in division problems ask them to give an estimate of the quotient and then discuss the process and the difference between their values of the given estimate and the actual quotient, particularly if there is a significant difference between the two. This type of discussion of the difference between an estimate of the quotient and the actual calculation of the quotient helps students to understand the division concept much better. It also prepares the students for application of division. After discussing the estimate one should give a related problem (with slightly changed numbers) and again ask for the estimate first and the actual calculation later.
mate first and the actual calculation later.

For example, let us assume that the first problem discussed was
$982 \div 45$
then the changed problem could be $9820 \div 450$
or $9820 \div 45$.

Students should be asked to give the estimate of these later problems and reasoning behind the new estimates. The teacher should also ask the students in what ways the new problem is different than the original problem, in what ways it is similar and in what ways the estimate for the new problem is different than the estimate of the original problem. What it the relation between the two estimates. Is there any pattern emerging? Ask them to give the reasoning for every answer they provide. Ask the students to write the estimates and then answer the problem by making actual calculations. Now ask them to compare the two answers. If there is any discrepancy between the estimate and the answer obtained by actual calculations, then ask why there is a difference between the two. Is the difference significant? What was the effect of the changed problem on estimation?

(6) Comparison of the product of the divisor and the quotient at each stage of the process with the dividend and the difference at the stage with the divisor.

D. Setting division patterns and the related multiplication patterns:

$\begin{array}{r} *** \\ 4\overline{)2000} \end{array}$ ◆ $4 \times *** = 2000$.

$400 \div 40 = 10$ ◆ $\begin{array}{r} ** \\ 40\overline{)400} \end{array}$ or
$40 \times 10 = 400$

The teacher should constantly change division problems into multiplication problems and vice-a-versa. Now do the same by changing the number in the division equations by using symbols and letters. For example,

$@ \div \$ = \&$ ◆ $\& \times \$ = @$
◆ $\$ \times \& = @$
◆ $@ \div \& = \$$
◆ $\begin{array}{r} \& \\ \$\overline{)@} \end{array}$
◆ $\begin{array}{r} \$ \\ \&\overline{)@} \end{array}$

and similarly change the following problem into corresponding multiplication and division problems.

$a \div b = c$. ◆ $\begin{array}{r} c \\ b\overline{)a} \end{array}$
◆ $\begin{array}{r} b \\ c\overline{)a} \end{array}$
◆ $a \div c = b$
◆ $b \times c = a$, and
◆ $c \times b = a$

Similar examples must be provided for students so that they can see the relationship between multiplication and division.

E. Language of mathematics.
Now the teacher should ask students to make up number stories relating to the arithmetical equations.

For example, Consider the equation

$\begin{array}{r} ? \\ 10\overline{)400} \end{array}$

Ask students to tell you a number story which uses number 10 and 400 and tells you to divide. An example of such a story could be:

> My mother bought 400 ornaments for our Christmas tree this year in 10 boxes. How many ornaments are there in each box, if each box contains same number of ornaments?

Development of multiexpressions for division concept and division procedure is another important activity during initial teaching. When a teacher uses only one language expression for a particular mathematics operation, we make a child "command specific" or "language specific." The objective of this activity is to prevent children from becoming 'language specific' or 'command specific.' Although, the repeated use of the same language expression for a particular mathematics concept provides a sense of security for most children, but at the same time they begin to rely only on that particular expression as the linguistic manifestation for that concept. When the child hears that particular command or that particular language expression which he is familar with, he is able to perform that operation but when the mathematics language expression in the problem is slightly different than the familiar one, the child finds it difficult.

More than ten percent of children at age 13 on several standardized examinations and studies could not give the correct answer for problems such as:

Find the product of 24 and 12.
(a) 36 (b) 12 (c) 288 (d) 2 (e) .5

Here, the reason was that the students were familiar with the operation of multiplication and the term "multiply," which was not used and they were not familiar with the term 'product.' This is an example of a child being command specific or language specific. Similarly, many children did not know what to do in a problem which asked them to find the quotient, although they did know how to divide. The reason for the difficulties in these problems is that those who were not exposed to the term 'product' or 'quotient' specifically did not know what the terms meant. In most textbooks the most popular command expressions for multiplication is the word multiply followed by problems and similarly in the case of division the command is the word divide followed by problems without realizing that their finding the product or the quotient as the case may be. They have become command specific when we use only one type of language models. For example, consider the following mathematical expressions:

(i) 4 + 6 is generally read as four plus six
(ii) 16 − 8 is generally read as sixteen minus eight
(iii) 7 x 8 is generally read as seven times eight
(iv) .8 ÷ .42 many children will read this as .8 divided by .42 but they write this expression as $.8\overline{)\,.42}$ and continue to read this as .8 divided by .42. An example of syntactic error.

Notice the difference in mathematical expressions from (i) to (iv). The first three expressions are direct sentences in active voice format and the fourth expression is in passive voice and is an example of indirect expression of this concept. As a result, the child continues to rely on the same syntactical and language model in (iv) as he was accustumed to in (i) through (iii). In (iv) when he encounters a different language model he does not really notice it and therefore does not see the contradiction between his written and spoken form of language for the operation. When we deal with whole numbers in lower grades the context of the numbers is able to help him to correct his error, because at this level we use a number as a dividend which is bigger than the divisor. But, later on the context of the numbers and the problem does not help the student and therefore the students continues to misread the problem. It is, therefore, extremely important that this is noticed early by the teacher and corrected before fifth grade otherwise it becomes much more difficult to correct. To avoid and prevent this difficulty, we should be reading all the expressions (i) to (iv) in several different, in direct and indirect, in active and passive forms, so that children do not become command and language specific. So that they begin to think in the mathematics language rather than just look at specific language expression and words in mathematics problems. For example:

(i) 4 + 6 can be read as: 4 plus 6; 4 increased by 6; 6 added to 4; 4 added by 6; 6 more than 4; 6 after 4; add 6 to 4; etc.
(ii) 16 − 8 can be read as: 8 subtracted from 16; subtract 8 from 16; 16 take away 8; 8 less than 16; 16 decreased by 8; 16 diminished by 8; 8 before 16; etc.
(iii) 7 x 8 can be read as: 7 times 8; 8 times 7; 7 groups of 8; 8 groups of 7; 7 multiplied by 8; 8 multiplied by 7; 7 added 8 times; 8 added 7 times; 8 added to itself 7 times; the product of 8 and 7; etc.
(iv) .8 ÷ .42 can be read as: .8 divided by .42; number of groups of .42 can be made from .8; .42 divides into .8; .42 goes into .8; ratio of .8 and .42; the quotient of .8 and .42; etc.

When children see a mathematics equation read in different mathematics language expressions, they begin to look at the language of mathematics rather than just chosing or focussing on specific words or phrases: They are not dependent just on clue words. This approach of emphasizing the multi-language expression usage in mathematics reduces children's anxiety regarding mathematics word problems.

E) Initial teaching of a concept vs remediation

While teaching division for the first time, the emphasis should be on the development of the concept of division by concrete and linguistic models. But, when we are remediating we have to develop not only the concept, but we also have to eliminate inefficient and faulty conceptual models that the student may have acquired. This is best done by asking examples to demonstrate a particular aspect of the concept and counter examples to show under what conditions a particular condition does or does not work. It is not enough to develop the model of the concept in question, but we should help the students to generalize the model to similar situations and also to discuss the limitations of the generalization of that model.

During the initial teaching, a teacher through concrete modeling and its representations for a concept should provide the child the opportunities to form the underlying mathematical schemas from which algorithms and methods of solving problems can be derived. When a student does not have the underlying mathematical and cognitive schemas then the child uses the algorithm or "recipe" without having no lasting connections with other concepts and algorithms. What we can learn with understanding depends on our currently available schemas. But when the understanding is developed it also develops new cognitive schemas. The learning of this kind is longer lasting and naturally alters the previously designed schemas.

G.) Division of decimals

$.53 \div .4$ ◆ $.4\overline{).53}$

$$\frac{53}{100} \div \frac{4}{10}$$

$$\frac{53}{100} \div \frac{40}{100}$$

$53 \div 40$ ◆ $40\overline{)53}$

H. Mechanistic (rote) understanding vs conceptual (cognitive) understanding

Skills acquired both by mechanistic/rote learning and with cognitive understanding create sensitivity on the part of the child to other related mathematics concepts. Rote skill makes the new learning efficient but accompanying cognitive skills makes the relationship between the new learning and old learning possible and meaningful. The rote learning helps in acquiring new concepts but the new learning acquired in this manner may not be long lasting because it is less adaptable. For successful learning and performance, the best combination is to learn the concept by the help of cognitive skills and then use rote learning techniques to automatize it. In many classroom situations, we have also observed the opposite combination working quite effectively. The key to effective teaching and learning is the development of the two. When the student has the cognitive understanding of a concept, but it has not been automatized, then during the applications of that concept he has to derive a fact or a procedure all over again, and therefore, in such situations the rate at which the solution is achieved is much slower. Whereas, when the student has acquired the mastery at a rote level, then the rate of solving a problem is faster but the student may not be able to initiate a solution if the problem is not completely familiar. Therefore, we need both types of mastery to understand mathematics concepts. One or the other is not enough. It is necessary but not sufficient.

Mechanistic understanding does not result into good applications whereas conceptual understanding can result into better applications. A student with mechanistic understanding is able to apply a recipe but he may not have the understanding for the concept. For example, if we ask students to solve the problem:

$.23\overline{).0598}$

Many students can solve the above problem mechanically by articulating the steps as: move the decimal to two places in both the dividend and the divisor and then divide in the usual way. But, if you ask him why does he move the decimal to the right in the dividend and divisor by equal number of places? Why does it work? What do we achieve by doing so? A student with machinistic understanding may not be able to answer these questions. He lacks conceptual understanding. He is not functioning at the "communications level" of understanding and therefore regression is not only possible but inevitable. [For detailed treatment of the topic levels of knowing see the **Math Notebook,** volume 6 (1&2).] These students neither can give a good estimate for the quotient correctly for this very problem nor can see the applications of these problems to real-life situations. As a result the learning of this procedure remains at a machinistic and rote level. They know it at the "recipe level." This clearly indicates lack of understanding of the concept. Although emphasizing understanding at the initial teaching level, in the opinion of many teachers is quite time consuming but it can save a lot of time in remediation processes. And at the same time it also creates better attitudes on the part of students towards mathematics. As such division is reviewed from fourth grade to high school level and even then we have students who do not understand what division means and how to use the division algorithm. Part of the reason is that enough appropriate conceptual models were not presented at the initial teaching level. Therefore, it is important to develop the concept of division effectively both at the understanding level and algorithm level.

I. Division as a Skill

Division algorithm is a skill (consisting of several steps and concepts) and therefore it has to be coached (after teaching the several concepts involved in the algorithm). And coaching requires two elements: First, the student should see examples of its perfect execution, perform the skill under supervised conditions until the understanding and mastery; Second, the feedback for reinforcement and for correcting an error should be given immediately. The objective of learning any skill is to learn it at an automatized level of proficiency. And that comes with effective coaching and proper practice. The best approach, for concepts such as division, that I have found to be very effective is:

Step 1: Write 25 multistep division problems on the chalkboard. Examples:
$12\overline{)592}$, $9\overline{)7895}$, $81\overline{)79564}$, etc.

Step 2: Ask each student in the class, in turn, to estimate the quotient for one problem from the above set. The first student estimates the quotient for one problem and so on. The important thing is that the teacher should ask a student a question that is appropriate to his ability. If the student has difficulty with certain numbers ask the question that you think is at the level of the student. Therefore, the problems written on the chalkboard should reflect the level of the class. There should be a range of difficulty in the selection of these 25 questions.

Step 3: Now ask the students to multiply the estimate with the divisor, one at a time.

Step 4: Ask them to compare the product with the component of the dividend under consideration, again one at a time.

Step 5: Now ask them to perform the subtraction operation in each problem, each student solving one problem from the set.

Step 6: Ask them to compare the difference obtained in the above step with the divisor in each problem to help them see that they have the appropriate remainder.

Step 7: Ask them to bring down the next number from the dividend to the new dividend and so on. This process is repeated for each problem.

Each student gets a chance to answer few questions or components of the algorithm at each step. Complete all the operations involved in the division algorithm in each of the 25 problems. Each student should get a turn and the errors must be corrected right then and there. For any information to transfer from short memory to long term memory an average person needs to see it, hear it, or manipulate it approximately from 18 to 25 times. The purpose of the above exercise is to achieve this. Therefore, this exercise requires good task analysis on the part of the teacher. (Breaking the division algorithm into its smallest of components and practicing each one of the components many times with each student.)

After completing each of the 25 problems, now take a new problem and complete all the steps for this one problem by going through the whole procedure. Now let individual students solve problems independently and you help them individually.

Homework must be assigned at this stage, but the teacher should make sure that the students have mastered the steps involved in the division algorithms in the class itself.

K. Classification

An important prerequisite for division is classification or grouping. When we consider the problem 12 ÷ 4, two important concepts are playing a crucial role: classification (grouping) and the language. To answer this question: How many groups of four can we make from 12?, the concept of classification is involved. To see the importance of language and the concept of classification, let us consider the following problem: 1/2 ÷ 1/4. We ask the question: How many fourths (1/4ths) are there in a half (1/2). Or how many groups of 1/4ths can we make from 1/2? Or, let us consider the following: a ÷ b. We ask how many groups of b's can we make from a? As we can see language plays an important role in conceptualizing the idea of division. There are many children who read

$$(1/2) \div (1/4)$$

as 1/4 divided by 1/2. There are two reasons for this error: One, the children did not understand the language behind the concept (as we already discussed in the section of language) Second, children see only the $\overline{)\quad}$ notation for division and they are not asked to translate " ÷ " into "$\overline{)\quad}$" and vice-a-versa. As a result, they have a limited symbolic model for the concept of division.

L. Role of short division

Most classroom teachers emphasize the long division algorithm and a very few teachers really pay enough attention to the short division process. As a result, we observe the following behavior. A child is faced with the following problem:

Reduce the following fraction into its lowest terms

$$\frac{525}{1200}$$

The child, rather than using the short division, actually works out two long division problems.

```
  105
5)525
  -5
-------
  02
   0
--------
   25
   25
---------
   00
```

```
  0240
5)1200
  10
-------
  020
   20
--------
   00
   00
---------
```

This kind of behavior is repeated even in simple fraction reduction and other related problems. It is very time consuming. If the short division is emphasized much of the work in fractions, decimals, percentages, etc. can be simplified for the child and their conceptualization will be greatly enhanced. Short division by easy numbers should be taught to all children. Not only it gives them an easy method, but it also helps them to conceptualize the idea much better. By practicing short division children's estimation skills also get better.

M. Role of calculators

Many mathematics educators are advocating that with the inexpensive availability of the calculator and computers it is not necessary to really emphasize the long division algorithm in elementary classrooms. Calculators and computers are very good tools. They can be very effective tools in mathematics conceptualization and in the development of skills. But, the key is how to effectively use them. The use of the calculators in the classroom should be for the following purposes: First, to develop basic computational skills; Second, for developing children's ability in problem solving. The use of calculators should not be encouraged in situations such as when the child either refuses to memorize the facts or has mistaken feeling that it is impossible to memorize the basic arithmetic facts. In such situations, the use of calculators is counter productive. One should find other strategies. (Our experience tells us that any child with average intelligence has the ability to memorize the basic facts whether he is learning disabled or not. Almost all children are able to master all the words from the many of the popular words lists and are able to memorize their spellings.) To replace the task of memorizing basic facts by the use calculators is not a good idea. We are not against the use of calculators in the classroom, but we want their use to be more effective and as a result we strongly advocate their use when the following conditions are met:

- The students using the calculator know their basic facts of addition, subtraction, multiplication, and division;
- The student is good in estimating the outcome of basic operations of addition, subtraction, multiplication, and division;
- The student knows the sequential steps involved in the algorithms of these basic operations; and

- The student uses the calculator for problems solving.

As a problem solving tool, the calculator is one of the greatest revolutions that has come to the classroom. As a matter of fact, with proper use of the calculator, even the estimation skills can be enhanced.

N. Estimation Revisited

In real life situations, in our encounter with numbers, we estimate more often than we actually calculate. My estimate is that we use estimation in more than 90% of the situations and actually calculate very rarely. Only in rare situations that we calculate to get an actual and exact answer. However, in an actual mathematics classroom, we ask children to make actual calculations almost 100% of the time. As a matter of fact, in most mathematics textbooks the topic of estimation is treated very sparingly. For example, only after the child has learned the algorithm on multiplication then the last exercise in the chapter on multiplication in the textbook generally is on estimation. In that exercise children are asked to estimate the answer without performing the actual multiplication procedure. But what the children do is that they actually multiply and then round off the answer and give an answer as the estimated answer. That is not the kind of estimation we are looking for. They should estimate before they calculate. We, therefore, consider estimation as an important skill. Although, many educators argue that the ability to calculate mentally and to estimate has been obsolesced by calculator technology. And according to some of them, the most efficient way to check the reasonableness of an answer derived by a machine is not to apply an estimation or mental calculation techniques but to continue to recalculate until a satisfactory answer is produced. In our view, it is counterproductive, in this manner children do not see the number relationships.

Certainly recalculation can be a very efficient procedure for validating the answers to problems. But how does a calculator decide that an answer is erroneous and should be recalculated? When a calculation is redone, how does the user know that this new answer is more satisfactory. A considerable knowledge of number relationships, mental calculation techiniques, and estimation methods is required. Unless a user has a well-developed number sense and has acquired a store of knowledge about common measurement referents (e.g. the height of a tall man is about 6 ft), a calculator cannot be used intelligently. We believe that estimation and mental computation activities develop thinking procedures that can help children make intelligent uses of the calculator. For this reason, we hope that estimating and calculating mentally will not be neglected because of the appearance of calculators in schools. Moreover, our experience tells us that children who are better in estimation are better in problem solving also. The reason being that good estimators not only check their procedures after completing the problem but they also check the outcome of the procedure. As a result they are able to see quickly whether a particular procedure is going to yield the answer or not. Calculators, thus properly used can be very effective tools, in learning mathematics otherwise the calculators will be a detrimental tool to learning. A few days ago, I was in the post office. I bought some stamps and was mailing some other articles. The bill came to $11.13. I gave the postal calculator that he could not give the change without calculating it on the calculator. He could not figure out the answer in his head. He said he could do it before the advent of the calculators, but not now. The calculator was about 10 steps away from his station. He went to the calculator, computed the amount and then gave me the change back. And the amount of change he gave me was wrong. Evidently he pressed the wrong keys. He entered $25.25 in the calculator. When he gave me the change he did not even question the amount. Although he did ask why I gave him $15.25 instead of $11.25. Still he gave me the wrong change. He was lost even with his calculator. He did not have good estimation skills. Calculators are here and they are here to stay, but we have to train children not only in their use but also in their correct use. And key skill involved in this is estimation. Without estimation skills being emphasized we might face a major problem created by calculators.

Having said this let us see why we consider the teaching of long division as an important aspect of mathematics teaching and curriculum.

Compare the following examples:

$54\overline{)24536}$ and $x + 4\overline{)24x + 6x + 19}$

Compare the steps and processes involved in these problems. The basic steps involved in the two problems are almost the same. Moreover, the understanding of the steps in the arithmetic problem helps a student to see the division problem with polynomials. Therefore, the experience with long division problems is very beneficial in understanding and conceptualizing the long division of polynomials and other algebraic expressions.

Long division is a very good example of the integration of the basic operations learned so far and in the understanding of the arithmetic concepts concerned.

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THE JAPANESE SOROBAN: A TOOL THAT ENHANCES INSTRUCTION THROUGH CONCRETE LEARNING

By Frances R. Metallo
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Instructors of arithmetic and remedial mathematics are always faced with the challenge of trying to make math appealing for their students. How can we as educators make basic mathematics interesting? How can we get our students to acquire a deeper appreciation of mathematics while learning concepts?

There are many answers and approaches to these questions. As an instructor at Jersey City State College, I have utilized in my teaching the abacus, specifically the Japanese abacus, with great success. I expose students to some of the history and the development of the Japanese abacus in addition to information about the special schooling required in Japan for operating the instrument with proficiency. I then illustrate the concepts and methods concretely. This adds excitement to the class and also motivates the students. I shall address each of these topics in detail. Also, I shall include some applications I use in my classroom, and I shall conclude with the impact of the abacus on Japanese society and education.

Unfortunately, the abacus is viewed by many Americans as a toy and a relic instrument (Adkins, 1956; Flewelling, 1960). However, in Oriental countries, such as Japan, the abacus is part of the culture in business and in education.

The Japanese word for abacus is "soroban" (Adkins, 1956; Smith, 1958). The soroban was imported from China during the sixteenth century (Adkins, 1956; Smith, 1958). The Chinese abacus has seven beads per rod: two beads above the crossbar and five beads below the crossbar (see Figure 1).

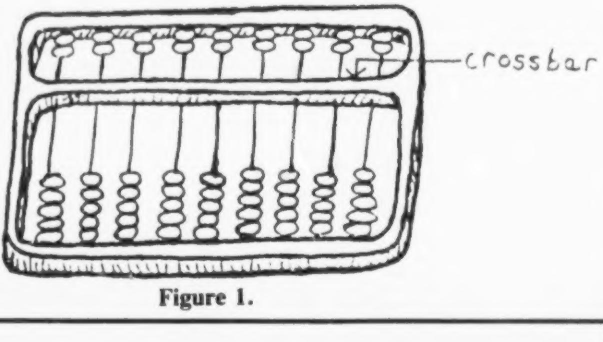


Figure 1.

Figure 1. The Chinese abacus consists of seven beads per rod. Two beads are above the crossbar and five beads are below the crossbar.

As the soroban became popular in Japan it was studied extensively by mathematicians. Improvements, both in form and operational techniques, led to the present day soroban.

Towards the close of the nineteenth century, the Japanese modified the Chinese form of abacus by eliminating one bead from above the crossbar (Kojima, 1954, 1963). This new form of the soroban was called the Traditional Abacus (see Figure 2).

Around 1920, the Modern Japanese Abacus replaced the Traditional Abacus (Kojima, 1954). This form of soroban consisted of five beads per rod. One bead had been eliminated from below the crossbar (see Figure 3) to create this modern version.

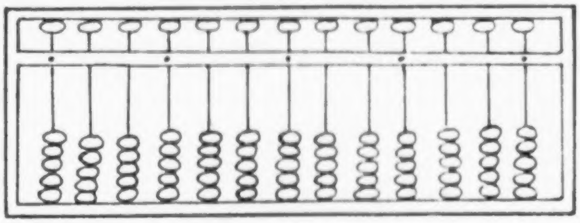


Figure 2. The Traditional Japanese Abacus which was modified from the Chinese Abacus near the end of the nineteenth century consisted of six beads per rod.

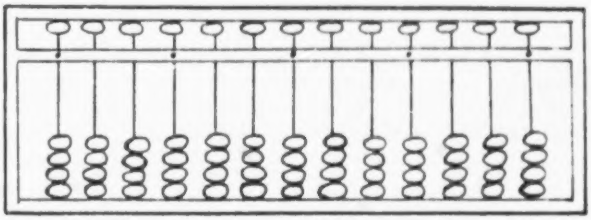


Figure 3. The Modern Japanese Abacus which replaced the Traditional Japanese Abacus consisted of five beads per rod: one bead above the crossbar and four beads below the crossbar.

To operate the soroban with speed and accuracy requires technical skill. In 1946, this was demonstrated in a contest between the soroban and an electric calculator sponsored by the United States Army newspaper, **Stars and Stripes** ("Addition," 1964; Kojima, 1954; "The Centuries-Old," 1947). The American representative using the electric calculator, was Private Thomas Nathan Wood, a clerk in the Army Finance; and representing Japan, using the soroban, was Kiyoshi Matsuzaki, a clerk for the Japanese Communication Ministry (Kojima, 1954).

The contest consisted of five events: addition, subtraction, multiplication, division and composite (Kojima, 1954). Each event had five problems except for the composite event which consisted of ten problems (Kojima, 1954).

Matsuzaki was victorious in all events except the multiplication, which requires many hand motions on the soroban (Kojima, 1954; "The Centuries-Old," 1947). All the events were timed under two and one-half minutes (Kojima, 1954). Matsuzaki was called "The Hands" because of his lightening speed ("The Centuries-Old," 1947).

As demonstrated, techinical skill is needed to operate the soroban. In 1938, the Japanese Ministry of Education required that students begin to receive soroban training as part of their math curriculum in the third grade (Brown, 1987; Lubarsky, 1986). To become proficient, students later attend private abacus schools called "jukus" (Brown, 1987). There are at least 60,000 jukus in Japan (Lubarsky, 1986). Japanese parents feel that sending their children to jukus will help them get jobs and do well in business in addition to the developing of mental calculation and concentration (Lubarsky, 1986).

Once an individual feels proficient, special examinations are given to measure the level of expertise. The All-Japan Federation of Abacus Operators, under the sponsorship of the Japan Chamber of Commers and Industry, administers the tests, which range from grade 3 to grade 1, with grade 1 representing the highest level of expertise (Kojima, 1954). Examinations are administered three times a year, and the individual is certified that particular level if he passes the required examination (Kojima, 1954).

Each grade consists of three separate tests: twenty multiplication; twenty division; and ten addition and subtraction problems (Lubarsky, 1986). As the grade level increases, the tests become more difficult. For example, in comparing an addition and subtraction problem of grade 3 to grade 1, both levels have fifteen numbers; however, the third grade level has three to six digits as opposed to the first grade level, which has five to ten digits (Kojima, 1963).

According to Kojima (1963), each test has a ten minute time limit. The accuracy required for the first grade level is 80%, whereas for the second and the third grade levels a 70% accuracy is required (Kojima, 1963). As can be seen both speed and accuracy are essential to be successful.

In the commercial high schools, the soroban is a required subject. All students must acquire at least the second grade level in order to graduate (Kojima, 1963; Lubarsky, 1986).

Once the soroban is mastered many Japanese then do mental calculations. Mental calculation is a method which consists of mentally visualizing the abacus and performng the indicated problem by the usual soroban methods but on the imaginery instrument (Kojima, 1954; "Mathematics'," 1983). The Japanese word for this mental calculation is "anzan" (Kojima, 1954; Lubarsky, 1986). The amazing speed of mental calculation has also been demonstrated.

In 1952, at the Sixth All-Japan Abacus Contest, Yoshio Kojima added ten numbers which were ten digits each in 13.6 seconds (Kojima, 1954)..

A more recent demonstration of anzan was done by a junior high school student, Yuka Hatano, the champion at the annual competition sponsored by the League for Soroban Education of Japan in 1985 (Lubarsky, 1986). Yuka can add a column of twenty ten digit numbers in her head (Lubarsky, 1986). She adds and subtracts as fast as the numbers are read and dictated to her from a page (Lubarsky, 1986). Yuka keeps the subtotals in her head and once the dictation is completed, she will have the correct answer (Lubarsky, 1986).

So far we have followed the history of the soroban, the schooling and training required to become a proficient abacist and its impact on society.

Educationally the soroban provides many opportunities for students. Before illustrating some of the applications for the classroom it is essential to become acquainted with the construction of the soroban.

Figure 4 shows that the soroban is divided by a crossbar (Kojima, 1954; Lubarsky, 1986), consists of five beads; one 5-unit bead and four 1-unit beads (Japan Chamber of Commerce and Industry, 1969; Kojima, 1954). The total for each rod is nine. The value of each of the beads is determined by the positions, which are based on the decimal system (Kojima, 1954). The beads below the crossbar represent a single decimal unit. Each bead above the crossbar represents five times the value of its position. For instance, a 5-unit bead on the units' rod would represent 5, on the tens' rod would represent 50, etc.

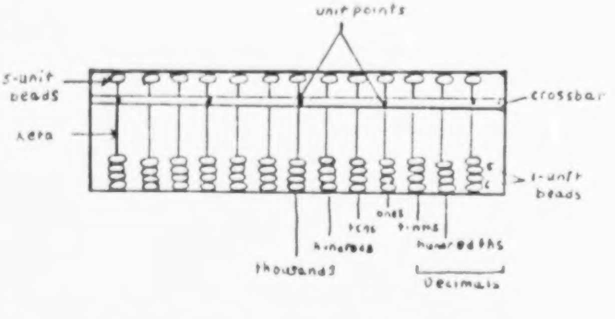


Figure 4. The construction of the Japanese soroban.

As illustrated in Figure 4, a unit point is at every third rod (Japan Chamber of Commerce and industry, 1969). The unit points are used to indicate the units (ones) place or to represent a decimal point (Kojima, 1954). Any unit point may be designated as the units' rod (Kojima, 1954). Once the unit point is selected, the rods to the left of the units' rod represent the tens, hundred, thousands, respectively — that is, 10^1, 10^2, 10^3, respectively — (Kojima, 1954). Similiarly, to the right of the units' rod are represented the tenths, hundredths, thousandths, respectively — that is, 10^{-1}, 10^{-2}, 10^{-3}, respectively — (Kojima, 1954). The soroban clearly illustrates place value, and that the value of a number depends upon its position.

To obtain value in any column, the beads are pushed to the crossbar, and the value it possesses depends upon its position (Kojima, 1954). Likewise, when the beads are pushed away from the crossbar, they lose value (Kojima, 1954). The code for the bead movement, which I will use for all of my figures, is as follows:

○ bead is in the original position and does not have numerical value.

● bead which has numerical value

Figure 5 illustrates how the numbers from zero to ten are formed on the soroban.

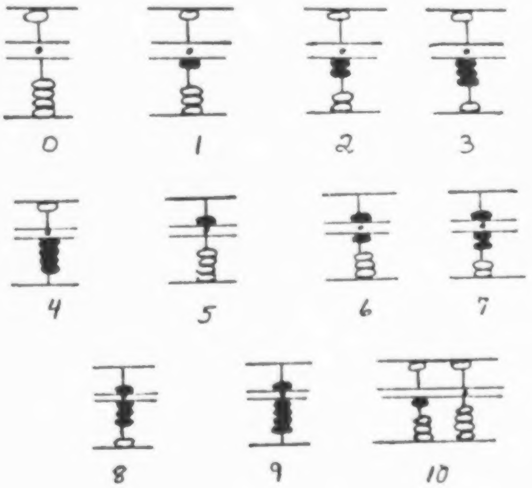


Figure 5. The numbers from zero to ten as represented on the soroban.

To represent zero on a rod, the 5-unit bead and all of the 1-unit beads are away from the crossbar as illustrated in Figure 5.

The soroban harmonizes with the normal way of reading an writing of numbers, which makes place value easy to see. I shall illustrate on the soroban the place value of given numbers and then use the concept of place value to write numbers in expanded form. I shall then conclude with word names.

For example, Figure 6 illustrates how 324 is represented on the soroban and that 324 can be written in expanded form as:

$324 = 3 \times 100 + 2 \times 10 + 4 \times 1 = 300 + 20 + 4$

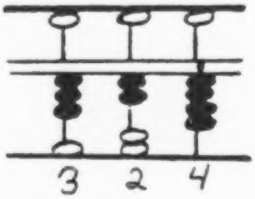


Figure 6. The representation of 324 on the soroban.

because 324 represents three hundreds, two tens, and four ones. The word name here is three hundred twenty-four.

Additional examples, expressing place value, expanded notation and word names, appear in Figure 7 and Figure 8.

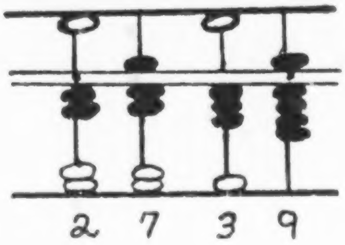


Figure 7. The representation of 2739 on the soroban plus its expanded form and word name.

expanded notation:

$2739 = 2 \times 1000 + 7 \times 100 + 3 \times 10 + 9 \times 1 = 2000 + 700 + 30 + 9$

word name:

two thousand, seven hundred thirty-nine

Figure 7 illustrates that 2739 represents two thousands, seven hundreds, three tens, and nine ones. Likewise, Figure 8 illustrates that 35,846 represents three ten thousands, five thousands, eight hundreds, four tens, and six ones.

The soroban easily demonstrates that zero is a place holder. This is particularily useful since many students have difficulty understanding this concept. For example, with the number 2403, zero holds a distinct place value, the tens. This is easily visible on the soroban (see Figure 9).

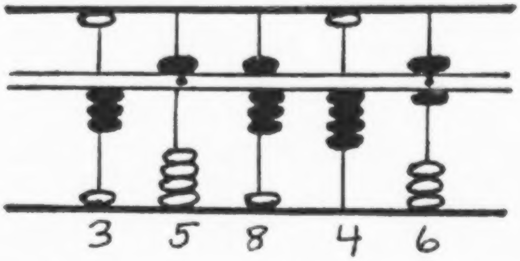


Figure 8. The representation of 35,846 on the soroban including its expanded notation and word name.

expanded notation:

$35{,}846 = 3 \times 10000 + 5 \times 1000 + 8 \times 100 + 4 \times 10 + 6 \times 1 = 30000 + 5000 + 800 + 40 + 6$

word name:

thirty-five thousand, eight hundred forty-six

Decimals can also be represented on the soroban: expressing place value, expanded notation, and word names. To illustrate decimals on the soroban, a unit point must be selected. This point represents the decimal point and the rod is designated as the units rod. I will use a ▲ to indicate the unit point selected to represent the decimal point as well as the units rod.

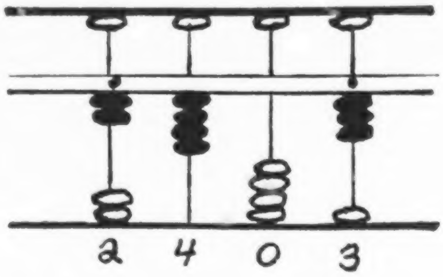


Figure 9. Illustration of zero as a distinct place holder using the number 2403 as set on the soroban. Here zero is in the tens place.

For example, Figure 10 illustrates how 473.68 is represented on the soroban and that 473.68 can be written in expanded form as:

$$473.68 = 4 \times 100 + 7 \times 10 + 3 \times 1 + 6 \times 1/10 + 8 \times 1/100 = 400 + 70 + 3 + 6/10 + 8/100$$

because 473.68 represents four hundreds, seven tens, three ones, six tenths, and eight hundredths. The word name here is four hundred seventy-three and sixty-eight hundredths.

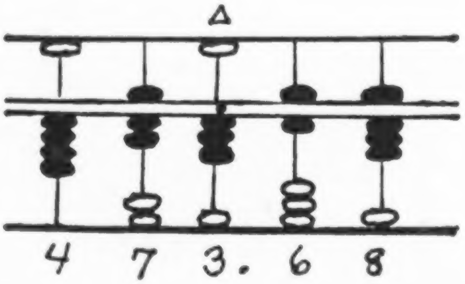


Figure 10. The representation of 473.68 on the soroban.

Additional examples, expressing decimals appear in Figure 11 and Figure 12.

expanded notation:

$$35.192 = 3 \times 10 + 5 \times 1 \times 1/10 + 9 \times 1/100 + 2 \times 1/1000 = 30 + 5 + 1/10 + 9/100 + 2/1000$$

word name:

thirty-five and one hundred ninety-two thousandths

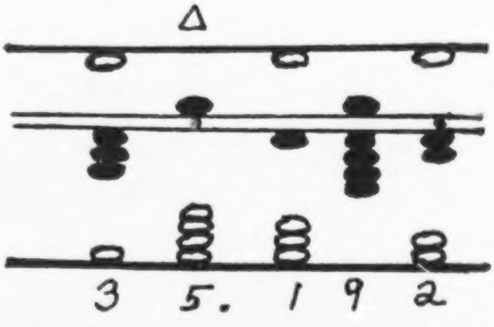


Figure 11. The representation of the decimal 35.192 on the soroban, plus its expanded form and word name.

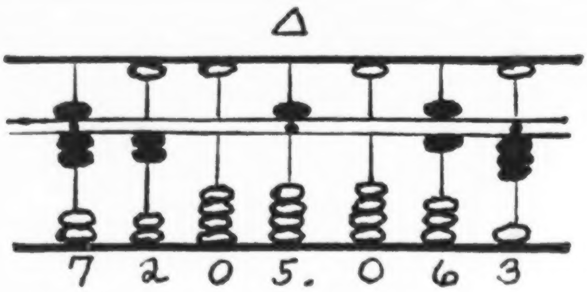


Figure 12. The representation of 7205.063 on the soroban including its expanded notation and word name.

expanded notation:

$$\begin{aligned}7205.063 &= 7 \times 1000 + 2 \times 100 + 0 \times 10 + 5 \times 1 + 0 \times 1/10 + 6 \times 1/100 + 3 \times 1/1000\\ &= 7000 + 200 + 0 + 5 + 0/10 + 6/100 + 3/1000\\ &= 7000 + 200 + 5 + 6/100 + 3/1000\end{aligned}$$

word name:

seven thousand, two hundred five and sixty-three thousandths

Figure 11 illustrates that 35.192 represents three tens, five ones, one tenths, nine hundredths, and two thousandths. Likewise, Figure 12 illustrates that 7205.063 represents seven thousands, two hundreds, zero tens, five ones, zero tenths, six hundredths and three thousandths.

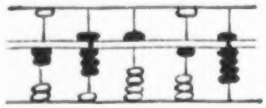
The following exercise can be used in the classroom to further reinforce the previous concepts discussed.

Exercise 1

A. For each of the following write:

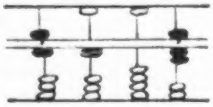
a) the correct number under each picture,
b) the expanded notation, and
c) its word name

1)



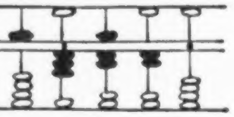
a)____________________
b)____________________
c)____________________

2)



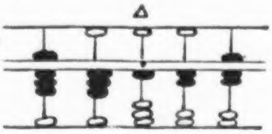
a)____________________
b)____________________
c)____________________

3)



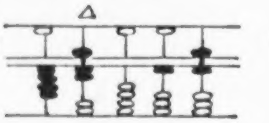
a)____________________
b)____________________
c)____________________

4)



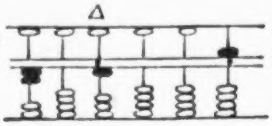
a)____________________
b)____________________
c)____________________

5)



a)____________________
b)____________________
c)____________________

6)



a)____________________
b)____________________
c)____________________

Solutions to Exercise 1

1. a) 28,519
   b) 20,000 + 8,000 + 500 + 10 + 9
   c) twenty-eight thousand, five hundred nineteen
2. a) 6,107
   b) 6,000 + 100 + 7
   c) six thousand, one hundred seven
3. a) 53,720
   b) 50,000 + 3,000 + 700 + 20
   c) fifty-three thousand, seven hundred twenty
4. a) 831.27
   b) 800 + 30 + 1 + 2/10 + 7/100
   c) eight hundred thirty-one and twenty-seven hundredths
5. a) 47.016
   b) 40 + 7 + 1/100 + 6/1000
   c) forty-seven and sixteen thousandths
6. a) 201.005
   b) 200 + 1 + 5/1000
   c) two hundred one and five thousandths

The soroban demonstrates many concepts besides those already presented. The soroban can perform all the basic operations (addition, subtraction, multiplication, and division) of whole numers and decimals (Kojima, 1954). Calculations involving more than one unit of measurement — such as feet and inches or pounds and ounces or hours, minutes and seconds — can be done on the sorobam as well as the extraction of both the square roots and cube roots (Kojima, 1963).

It is evident that the soroban requires skill to operate. Being successful on the soroban requires understanding and applying rules for bead manipulation and fingering, knowing the compliments of 5 and 10 in order to perform the abacus calculations, and performing calculations from left to right.

The soroban is very much a part of the Japanese culture. For example, in banks and in other financial institutions, such as insurance firms and trading houses, the soroban is still being used because it has been discovered that the soroban is more accurate (Simons, 1983).

In the educational setting, using the soroban enables students to see concepts concretely (Mizui, 1986). The soroban bridges the gap between the concrete and mathematical symbolism for students (Brown, 1987; Imada, 1985). The soroban helps students to think clearly and to see the relationship between the numbers and the concepts (Brown, 1987; Mizui, 1986; Simons, 1983).

There is also evidence that the use of the soroban helps to develop listening and concentration skills (Imada, 1985; Mizui, 1986). Furthermore, it is now believed that the soroban is the main reason why school children in Japan consistently score higher on the standardized math achievement tests (Brown, 1987; Imada, 1985; Lubarsky, 1986).

The use of the soroban in the classroom does have a positive effect on students if introduced properly. The purpose and to what extent any manipulative device is used depends upon the class, time and teacher. However, using the soroban, as I have, enhances the learning situation. The students not only saw some concepts concretely, they also saw the development and the study of math in another country, thus developing necessary skills and broading their intercultural awareness.

Making the learning of mathematics more enjoyable both helps students retain the concepts and causes them to feel more positive about mathematics. Students in arithmetic and remedial math classes need to have some positive experiences for future success in mathematics. Therefore, if using the soroban helps to develop that positive attitude, then such use has served a very important purpose.

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Ontogeny takes over Phylogeny : How Do We Learn Arithmetic?

(Mahesh C. Sharma)

A child counts a collection of objects and announces the count by calling out a number; another child makes some tally marks on a paper or slate corresponding to the count of a collection of objects; yet another child directly writes the number (in the socially accepted symbols) for the count of a given set of objects. These are a few examples of young children's early quantitative behaviors. It is a familiar sequence of arithmetic behavior in any early childhood or an elementary school classroom. These children are performing arithmetic operations almost in the same way as their ancestors did many hundreds or thousands of years ago. Does it say something about our development in the context of number sense and quantity? Does it show some relationship between the way a child acquires number concepts and how its ancestors acquired the same concept thousands of years ago? Close observation of children's early activities suggest that there is a relationship. To show that relationship is the objective of this issue.

This all too familiar an observation demonstrates an important phenomenon: "Ontogeny recapitulates phylogeny - that is, development within an individual (ontogeny) will both mirror and incorporate the development of the specie (phylogeny)" it belongs to. (Hughes, 1986; p. 78) This phenomenon has been observed at several levels. The development of an organism proceeds on several mutually dependent yet parallel continua: physico-biological, cognitive-intellectual, and conceptual-mathematical (as described here).

The existence of this phenomenon at the physical and biological level and its explanation is mostly accepted (although the debate, with some intensity, among biologists still goes on). The information that in biological terms an individual member of a particular specie repeats the life span of the whole specie in its individual life, but in a much shorter period of time already forms part of any educated person's common knowledge base now. For example, each organism begins with a cell and then goes through almost the same developmental steps as the whole specie has gone through. This is at a physical, biological level. It is true for all species and therefore it is true for human beings also. The same process of biological development is repeated by each human being. Each human child repeats the same developmental stages of the human specie in a very, very short period of time; the benefit of evolution. The acquisition of physical skills and maturation milestones takes place by the individual human child almost on the same predictable lines and sequence of the development of our specie. This is at the gross physical level and in general it proceeds on the expected developmental lines. But, in a few specific cases, whenever, a small group of a particular specie separates from the general main stream of life, it begins to develop according to a new developmental path, because it's individual experiences become different than the experiences of the mainstream. Nature is full of these interesting anamolies. The development of this kind of anamolie at the gross level takes probably thousands of years if not millions of years. But even in these tangential developments the individual members of the specie also follow mostly the same developmental sequence of this group's general development.

A similar process of 'ontogeny taking over phylogeny' is again demonstrated by the repetition of cognitive and neurological milestones by an individual human being. The human specie has gone through extensive cognitive evolution. The human specie has passed through several stages of intellectual development and we see that each individual member of the human specie also passes through the same or close to similar stages of cognitive development. This too reflects a general correspondence in the relationship between the development of a specie and its individual members. It does not mean that the intellectual development of the human race does not improve with this evolution. It does. The intellectual approach to problems has constantly improved over the history of the human race. The problem solving approaches and strategies have shown a continuous improvement with each generation.

The explanation of the existence of the phenomenon where ontogeny recapitulates phylogeny at the cognitive level is more recent. Piaget in his pioneering work in genetic psychology and epistemology showed that children's cognitive development follows certain patterns and these patterns mirror the pattern of human development. He says for example:

> "Any explanation of the child's developement must take into consideration two dimensions: an ontogenic dimension and a social dimension (in the sense of the transmission of the successive work of generations)."
> (Piaget & Inhelder, 1969; p. 157)

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Piaget, of course is referring here to the general cognitive development of children. His interest and study was confined to the study of patterns of cognitive development in children. He showed by painstaking observations that this development follows mostly along predictible lines. The debate whether it is continuous or discontinuous; is hierarchical or not; it is determined by cultural inputs or not, etc, continues. But the broad outlines of the patterns of development are evident and pretty much acceptable by most cognitive psychologists. He made a real contribution in increasing our understanding of children and their intellectual development. Which in turn has helped us in understanding the development of the whole human race.

Once again, anamolies in cognitive development are also evident when social and cultural experiences provided to a group of people have substantial differences in content and function. When the social and cultural patterns of a particular group are substantially different than the concomitant developmental patterns in cognitive development are also altered substantially. Although this difference in patterns of development may take a few generations as Piaget further explains:

> "...[A]n equilibrium is necessary to reconcile the roles of maturation, experience with objects, and social experience." (p. 159)

Following the analogy from these two cases, now we come to a more specific example where ontogeny, once again, seems to follow phylogeny. Apart from poor teaching, poor preparation of children for learning mathematical concepts, learning disabilities, and other popular reasons for children's mathematical difficulties, we propose another reason for children's arithmetical difficulty, particularly number difficulty. And that is as follows: if exposed to learning experiences related to number, every child passes through the similar stages of number development as the whole human specie has gone through. Of course, we are not suggesting that this phenomenon applies to the development of all mathematical thought in general or even all arithmetic concepts. We are focusing only on children's number conceptualization. Although this idea has been under consideration, but it has not been accepted as tenable hypothesis. For example, Hughes (1986) discusses it in his recent book, **Children and Number: Difficulties in Learning Mathematics,** in the context of the children's development of written mathematics, but concludes that

> "There are a number of fascinating parallels between the representations of young children and the systems developed by various other cultures. These parallels do not prove the idea that 'ontogeny recapitualtes phylogeny,' or suggest any general course which human evolution and development must inevitably follow: the evolution of number systems takes many forms and has developed into many backwaters and blind alleys. Rather, it seems that there are important similarities in the ways in which people of different situations cope with the problem of producing written representations of number and arithmetical operations." (p. 94)

As his statement "the problem of producing written representations of number and arithmetical operations," indicates, Hughes is considering the existence of this phenomenon and then rejecting it for the possible explanation of the development of written number system and number operations. In such a general case, we agree, the idea had less validity, but we are interested in a much more specific concept. Our hypothesis is limited to **children's number conceptualization and its acquisition and to the understanding and conceptualization of fundamental and rudimentary number operations.** That is, in the process of understanding and conceptualization of number and operations on numbers, children do prove the statement 'ontogeny recapitulates phylogeny' as we shall see in the following examples. The idea was hinted in Piaget's work on children's conceptualization of number and other related concepts as he pointed out, for example, when he said "interesting similarities and dissimilarities between the evolution of geometric thinking and the development of geometric ideas in the child."

We want to, in this article, closely look at the kind of developmental similarity children's number conceptualization at a much more specific level. When we study the history of the development of arithmetic ideas (development of the concept of quantity: number and its operations), we find that there are important milestones in the development of quantitative ideas. The human race in the process of developing arithmetical concepts paused at (or took longer time to discover) particular arithmetic concepts, procedures, or representations. Hughes himself suggests.

> "There are undoubtedly some clear paralles between the development of children's representations of number and the evolution of the various systems of notations used by other cultures. They are both marked by an early and pervasive use of one-to-one correspondence, a gradual replacement with more efficient cipherised systems, and only a very late use of symbols for representing addition, subtraction and equality." (p. 92)

Let us consider a few very specific examples to elucidate this point: the human race took a long time and after passing through several intermediate stages of development reached the point where it could represent numbers in the form of well defined and consistent symbols (numerals); the concept of place value, as we use it today, is quite recent — even Romans did not have a well defined concept of place value in their number system; all the familiar algorithms and procedures of arithmetic that we use today are also of recent origins; representation and operation of fractions; representation and operation with rational numbers; and many other ideas in arithmetic show a definite developmental history. It is the number conceptualization where we see the developmental history more clearly evident in children's work.

The experience of different cultures at the level of conceptualization of number and operations on them is quite common. It is on the formal and at the abstract level of arithmetic that the diversity in its development in different cultures is more pronounced. The development of ideas appear in diverse forms. And on this point we agree with Hughes and there he is right, when he says:

> "At the same time, there is sufficient diversity-both in the children's representation and in the cultural systems — to show that human development does not follow a unitary path, and that ontogeny does not inevitably recapitulate phylogeny. **We should instead be thinking more in terms of similar human responses to the problems of representing numerical information on paper. We should also not forget that the property of the medium itself, and the actions involved in using it, indoubtedly influence the possible solutions to this problem."** (emphasis added). (p. 93)

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And this is where the cultures had different experiences and children also have different 'responses to the problem of representing' the number. But as far as the initial, actual conceptualization of number and the sequence in which they conceptualize is concerned, children do recapitulate the collective human experience. Only when it comes to representation of number and operations on them at the abstract level do we begin to see diversions in human experience; the diversity if the role of cultures comes into play. This reflects the whole cultural milieu: the language, the development of script, the nature of recording experiences, and many other cultural artifacts. Since the development route and nature of language is different in different cultures the representation of number abstractly also took different routes in different cultures, but not the actual conceptualization of number. And this is what we are interested in.

A young child passes through the same stages of learning arithmetic concepts as the whole human race. Wherever the human race paused to understand an arithmetical concept, the human child also pauses at the same concept longer than he does at others.

To see the validity of this hypothesis, one has just to observe young children learning arithmetic ideas, one will find the presence of primitive strategies that children use in acquiring arithmetical ideas.

Graham Flegg (1984) in his book **Number: Their History and Meaning** says "the evidence of finger counting is so widespread that we are forced to conclude that it has been a universal practice." (p. 14) Flegg further suggests that "fingers may well have been used as a method for comparing sets of objects by people who were unable to count. The principle of one-to-one correspondence which underlies the use of fingers also underlies the most fundamental form of written representation,, that of tallying. Tallies, as we have already seen, appear very frequently in the spontaneous representation of young contemporary children and tallies at the same time have been found throughout the history as a precursor to represent the number much more symbolically as the children show it as a precursor to more abstract level of representation. Static marks-representation of quantity, are easier to learn than the symbolic or even pictorial representation of dynamic activities such as addition and subtraction. . . there is often a trade off between efficiency and comprehension." (p. 8)

Now let us examine the parallel that exists between how the human race, as a whole, learned number concepts (and related rudimentary arithmetic concepts) and how contemporary children acquire and learn rudimentary arithmetic concepts. Let us observe the following examples closely:

1) Human beings used concrete objects to denote the count of other concrete objects and one-to-one correspondence to use that count. And even today, in many villages in India and Africa and some primitive societies throughout the world, people still use concrete objects to denote the count of other concrete objects. Similarly, very young contemporary children also show this behavior quite consistently. They learn the counting process by the same method: counting actual physical objects. And this continues to be the case in prekindergarten, kindergarten, and even in first grade classrooms. Even at this level, children actually use concrete objects to denote the count and the number they want to emphasize.

2) Human race progressed from actual physical, concrete representation of count (and therefore number) to representation of this number by tallies on walls as many cave dwellings of that time show. In villages in India and other Asian and African countries people still use markers (tallies) on the walls for keeping track of counts. "Tallies are used in Britain even today in many situations where we need to keep track of events (such as a batman's runs in cricket), or to make an ongoing survey of specific occurrances (different kinds of vehicles on a particular stretch of road, for instance). (Hughes; p. 84) And when children first learn the concept of counting, they use the same type of markers on paper to keep track of the count. Although the instruments of this process have changed over the course of history, the underlying process has not. They used to put these tallies on cave-walls, sand; they used to put notches on wood, bones, and stones; they used to place these on barks, animal skin surfaces, parchment papers; and now instead of cave-walls, people in villages (and children in modern classrooms) put the tallies on slates and paper. And in the very near future probably they will represent numbers as tallies as pacman on video screens.

3) The concepts of place value, fractions, signed numbers, and other derived secondary arithmetic concepts are difficult for most children in all cultures and countries. This is again, because, the human race took a long time to actually conceptualize these difficult and complex concepts. To understand, they paused at each one of these concepts for a very very long time. These are some of the very important "milestones" in the historical development of arithmetical ideas. Therefore, each child pauses at each one of these concepts slightly longer than they do on other arithmetial concepts. And they need to.

"The evolution of numerals thus followed a very similar pattern to that seen in the representations of quantity made by contemporary children....Children's early attempts are based on one-to-one correspondence (classed as iconic and pictographic), and these are replaced as they grow older by more cipherised (symbolic) methods of representation, in which each quantity is represented by a separate symbol. The advantage of the later-both for the children and for our ancestors-is that they are more efficient, and less cumbersome to write. However, this gain is accompanied by a corresponding loss in the ease with which they can be grasped by the uninitiated." (Hughes, p. 88)

Some linguists propose a template in the human brain which is possibly precoded for learning the rudiments of language. Whether that template is also "wired" for some number sense or there is a different one for number is an open question. But reason and extensive observations suggest, as almost all normal human beings, sooner or later, are able to acquire the number concepts that there might be a template in the human brain for number sense, which provides a predisposition for learning number or at least the number sense. And as in many other things, that human beings have learned, the number sense has evolved and it might now be part of the complex of predispositions and even reflexes suggesting some kind of template. On the other hand, the number operations are so new on the continuum of the development of human understanding of mathematics that it has to be formally learnt by each individual and, therefore, it has to be taught. It can not be acquired by itself by most people. Not at least, without the presence of appropriate learning experiences in the individual's environment. Therefore, it becomes all the more important to make sure that teachers pay attention to the existence and development of important "milestone" concepts in arithmetic and then in mathematics.

This hypothesis that 'ontogeny takes over phylogeny' at least in the conceptualization of number has serious implications to the teaching of arithmetic ideas to children in elementary classrooms and preschool experiences in homes and early childhood care centers. Arithmetic concepts are to be taught by keeping in mind the considerations how that concept was acquired by the human race. It took millions of years of experimentation and experience for the human specie to develop the concept of number, place value, fractions (remember Egyptians used fractions but did not have a good way of representing fractions, the current form of representing fractions is less than 1800 years old, the present form of representing signed numbers is not even 500 years old, the neat and compact forms of algorithms of basic operations are only a few hundred years old). In other words, there are important developmental milestones in the development of arithmetic and mathematics ideas. We can not teach arithmetic concept with the same methods and intensity; the method should vary according to the concept and the level of child's understanding of that concept and where the child is on the developmental sequence of that concept.

Since the human race has taken a longer time at each of these milestones, the individual member of the human race also takes slightly longer to conceptualize these particular concepts and passes through them, more or less, with the same type of activities as the human race used to understand the concepts. There are similar steps in the development of an arithmetical concept. They first got the idea at the **intuitive** level, then they explore it at the **concrete** level, then they represented it at the **pictorial** level and after a long time they were able to abstract it in the **formal abstract** form, the form which is familiar to us today. The same process should be followed when we are teaching children these concepts. A relatively long time has to be provided for the **free exploration** of the concept using diversity of concrete, manipulative materials, then teaching them the concept at the **concrete** level and then at the **pictorial** level, and then and only then should we bring it to the **abstract** level. All the earlier arithmetical concepts were developed by concrete models. The human race overcame hurdles in the development of arithmetic ideas by the help of concrete models. There is a definite lesson in knowing how human race solved its quantitative problems. The notion that the children will be able to learn a concept easily at the abstract level and there is no need to spend a lot of time on extensive concrete work is wrong. Yes, a minority of children may be able to learn a concept at the abstract level itself, but in the majority of cases it is imperative to follow the sequence of **intuitive-concrete-pictorial-abstract** representation.

Teachers of young children should always keep in mind that the places in arithmetic concept development, where human race has taken a long time the individual child will also take longer to conceptualize a concept. The mathematics curriculum development and teaching processes should reflect this information.

As teachers and parents, we are always anxious to teach the finished mathematics rather than the process of how mathematics was evolved. Both are extremely important when we think of long term implications of teaching. Evariste Galois, the young French mathematics genius who died at the age of twenty-one, in his last letter, before the fatal duel, stressed how the real process of discovery is different from what finally appears in print as the process of a mathematical proof. We are anxious to teach this finished proof, but not the process. For young children the process is more important than the finished product. "There is little doubt that the system we currently use is an extremely efficient and powerful tool, the result of a long period of development. However, this increase in power and efficiency has been achieved by incorporating features — such as fully cipherised numerals, place value and operator signs — which do not feature naturally in the representations of young children." They do not "feature naturally" in the form of 'finished product' that we formally see. In the formal teaching situations we want children to demonstrate the 'finished product' in the short span of time. We want children to 'develop' the idea in the sequence and form that is efficient. They would develop it if we give sufficient time and exploration to follow the natural sequence. Since the child is not given sufficient time to explore, many children have difficulty with arithmetic concepts. "As a result, we would expect that the highly condensed way in which addition and subtraction is presented (e.g., '3 + 4 = 7') will present considerable difficulties for young children." (Hughes, 1986; p. 94) This rushing to the formal and abstract without exploration may cause many difficulties. The teachers should use techniques of helping children pass through the natural sequence by creating the same type experiences.

Many times teachers do complain that it is not possible to give the amount of time necessary for 'free exploration' and also arrive at the 'finished product' in the alloted time in regular classroom situation. The complaint is genuine. But there is no way out of it. If we do not do it then we will pay a heavier price in time and failure later on when we will have to provide this kind of time in remediation. There is no simple substitute to it. The time is important but the most important thing is the sequence; the child must explore the numbers first at the concrete level (under the direction and guidance of an adult in a rich and diverse environment); then help to explore the concept at pictorial (representation) level and then and only then should the child be exposed to abstract and formal representations of numbers and its operations. (Sharma, 1987)

Teachers have a major role, but a more enlightened one, to play in this venture. An elementary school teacher should have a good understanding of the history of arithmetic ideas so that s/he knows the sequence of the development of these ideas. Although, as a whole race we have traversed far and wide in the field of mathematics from our earlier understanding of number. But, at the highest level of understanding of mathematics, it is done only by a small group of people in any generation. The bulk of the human race, at any given time, remains at a primitive level of understanding of arithmetic ideas. Although, our problem solving abilities improve with every generation, the initial understanding of arithmetic concepts by each generation is done by the help of primitive methods. Most people do not go beyond the rudimentary level of understanding of arithmetical ideas. And they are ready to accept and admit the existence of limited mastery in this without any hesitation. Even when the understanding is there, many times we use many of the same methods in learning number as our ancestors used. As a result, a clear understanding, on the part of elementary teachers, of the development of milestones in arithmetic concepts in particular and related mathematics concepts is important. This understanding will help the teacher to plan and provide more concrete experiences for the concept where human race paused for a longer time. When we examine the teaching of mathematics from this perspective it becomes quite clear that the teacher must devote more time on these milestone concepts. And use a lot more concrete materials to teach those concepts until the child has reached the stage when it is possible to see things abstractly. "The idea that a purely abstract mark on paper can represent a change or alteration of some kind does not, it seems, come at all easily." (Hughes, 1986; p. 93)

The development of formal arithmetic concepts is not uniform in all cultures. Some cultures have not yet developed the arithmetic concepts to the extent that we are accustomed to. In some cultures, the extent of number understanding is limited to a few numbers and concepts. Aberrations in learning arithmetic concepts exist at all times and in all cultures, but they could be explained in the similar manner as we used for physical and cognitive developmental anamolies. In those cultures where the experiences relating to the quantity are limited, the whole social groups may not develop the facility in quantitative concepts. The same thing can be seen, in many instances, when an individual even in a culture where the use of number is quite prominent, may not develop quantitative ideas because his/her experience base in number related experiences is quite limited. An individual may also have a limited acquisition of number concepts when the individual's quantitative experiences are not in synchrony with his general cognitive development. As a result an individual may not have a normal development in arithmetic and mathematical concepts. This may happen in any culture, some individuals do not develop according to the milestones of development because of their limited experience base and other individual abnormalities and specific handicapping conditions.

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Mathematics Culture

(Mahesh C. Sharma)

Children begin to read early and develop an interest in reading when they live in an environment where people read, where reading is used in daily living, and where reading is prized and appreciated. When a child is read to; when a child sees books and other reading materials his/her curiosity is heightened and when this curiosity is responded to positively, the child becomes a reader and acquires a valuable skill for successful living. In other words, in most situations, reading is done where there exists a "reading culture." The same thing applies to mathematics. In schools and homes where there is a "mathematics culture," children learn mathematics, children "do" mathematics, children understand and apply mathematics, and they take an interest in mathematics related activities. The mathematics culture heightens children's curiosity and interest in exploring quantitative and spatial relations in their environment. In other words, they naturally begin to explore and, therefore, learn mathematics. Unless this interest is "killed" they will learn mathematics with positive attitudes and will not be "afraid" of mathematics. To be motivated to learn mathematics and in turn to do well in mathematics, a child needs to be provided a mathematically rich environment and encouragement to realize his/her potential.

Many schools and homes lack this environment; they lack this mathematics culture. And they also do not pay enough attention to the development of such a mathematics culture. On the other hand, there are schools where this culture flourishes. There are schools in the most dilapidated of areas of cities where children travel for more than two hours to attend these schools because there exists a culture of achievement in areas such as mathematics and science. The mathematics culture is curriculum, teachers, parents, other children, but above else it is an attitude which fosters and encourages mathematics learning. There are both intrinsic and extrinsic factors that promote it. The mathematics curriculum in a school should not just be a course of study and sequence of courses and the description of those courses. A mathematics curricula that does not emphasize the inculcation of "mathematics culture" does not promote optimal mathematics learning on the part of children. If a child has normal growth patterns, i.e., there are no severe neurological or other debilitating handicaps, he/she will read and learn mathematics when there is an encourageing learning environment; when they live in and experience a reading and mathematics culture.

What is a mathematics culture?

It is an environment; it is an attitude. Mathematics culture is the feeling provided by adults to children that learning mathematics is fun and learning mathematics is natural. It is the message given by adults to children that learning mathematics involves solving real, meaningful problems, and it is the feeling that learning to solve problems using mathematical information and methods means to have successful lives. In such an environment curiosity about mathematics and achievement in mathematics is welcomed and responded to in positive terms. In other words:

1. It is the presence and development of a consciousness and understanding on the part of all (administrators, teachers, parents and children) that acquiring of mathematics information is an important factor in living successful and fruitful lives.

2. It is an awareness on the part of all people working with children that mathematics (quantitative and qualitative information) is an important factor and it provides skills in making decisions and solving useful and important problems not only in technical but also in non-technical situations and actual day-to-day living.

3. It is recognizing and seeing mathematics (quantity and spatial relations) in our lives and in the world around us. It is the understanding that mathematics is a human creation but it is also constantly around us both in nature and in things man-made. Learning mathematics is natural and it is abundantly present in nature. It becomes artificial only when it remains in math workbooks and we make no connections of it with the world around us.

4. It is an awareness that mathematics has played an important part in the development and evolution of our culture and mathematics is an important part of our complex of ideas in our culture. And as we respect and consider it important to impart through education the collective and individual human achievements in the fields of art, literature, music, science, history, and other fields of human endeavor we should also want to bring to the attention of children the human achievement in the area of mathematics also. Therefore, we want to impart mathematical knowledge with the same enthusiasm, vigor and interest and then would teach it in meaningful and inter-

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esting ways. That is possible only if adults themselves try to change their own attitudes about mathematics and mathematics related fields of study.

Where is it present?

The famous Polish born American mathematician, S. M. Ulam (1976) in his autobiography, **Adventures of a Mathematician,** writes

> "I had mathematical curiosity very early. My father had in his library a wonderful series of German paperbooks-**Reklam,** they were called. One was Euler's **Algebra**. I looked at it when I was perhaps ten or eleven, and it gave me a mysterious feeling. The symbols looked like magic signs; I wondered whether one day I could understand them. This probably contributed to the development of mathematical curiosity." (p.20)

These kinds of statements are made by many mathematicians and scientists. Almost all of them refer directly and indirectly to some kind of environment or encouragement either at home or at school. The effects of a positive environment begin very early. Because what a child receives from his envrionment plays such an important role in his mental growth, educational achievement, and his attitudes towards all learning. A child's home and school environment apart from the articulated curriculm also has a "hidden curriculum." This hidden curriculum sets the stage for forming attitudes towards a particular type of content, of learning. In a great many homes the children are read to, taught the names of things, and in general given a wide variety of experiences that are enormously helpful to them later in learning to read. But very few have an equally rich opportunity to absorb number ideas in the same way. In some homes, parents do take special interest in inculcating a mathematically conducing environment, and there children begin to get interested in mathematics. For example,

1. In the homes of most professional people, children see materials (books, magazines, and newspapers) that display numbers. This reading material deals with numbers and shows the interaction of numbers and quantitative ideas. In such situations parents automatically bring to the attention of their children the importance of mathematics in their own lives. There are definite steps these parents take to foster the development of number sense during the crucial years before and after a child goes to school. They take every possible opportunity of helping their children increase their thinking skills and to give them just as rich a background for analytical subjects like mathematics and science as they do for reading. They make conscious connections between these subjects.

2. In the homes of most professional people during formal and informal conversations and experiences children hear about the use of mathematics in their parents' lives, when they consciously and unconsciously share their decision making processes.

3. In the homes of most professional people, children are expected to do better not only in subjects of their choice but also other required subjects. The importance of knowledge in all spheres is emphasized and, therefore, the importance of mathematics learning.

4. In most professional people's homes and in most surburban schools and in some caring urban schools, children are constantly exposed to people who use and like mathematics in their professional careers, therefore, in such situations and environments, children are able to pick many more role models and identify with these role models who use mathematics in their lives. This identification by a child with a role model plays an important part in a child's life. As a matter of fact, this identification factor, at present, is playing an important role in the differential achievement in girls and boys. Currently, in our society, boys have many more role models to choose from, whereas, girls have fewer role models to choose from who use mathematics in their professional lives. Therefore, many girls do not realize, early on, the importance of mathematics in their future professional lives. As a result, girls begin to take less interest in subjects such as mathematics and sciences. That in turn limits their choices for colleges and professional fields. Only with conscious efforts can this be reversed.

5. In the homes of many immigrants, parents realize that education is the most important thing to acquire for successful living in their newly adopted country. They also realize that the first area where their children can excel without having a great proficiency in english is mathematics and, therefore, they impress upon their children that they should do well in mathematics. They also realize that in the current technologically oriented society proficiency in mathematics is a "passport" to success. Therefore, they talk to their children about the importance of mathematics and encourage achievement in mathematics. Many studies and our own observations attest to the fact that comparatively immigrant parents devote more time with their children on their children's mathematics homework and mathematics learning in general. In other words, they create a mathematics culture in their homes.

6. In many homes, where parents who themselves did not attain a great proficiency in mathematics but through their experiences they had realized that their lack of preparation in mathematics has hampered their progress and they do not want their children to repeat the stories of their own inadequacies. Therefore, they try to impress upon their children the importance of mathematics. They talk about the importance of mathematics to their children. They provide experiences to their children which will show them the importance of mathematics in real life. They talk about the people who have done well in life because they had done well in school, particularly in mathematics. They talk about these people with respect, so that their children will understand what they value and respect. This is a positive contribution towards creating a mathematics culture but also a learning environment.

7. Communities, where parades are held not only for the winning football team or for the local hero who has become professional football player but also for the winners in the international mathematics olympiad or for students who have scored perfect on the mathematics component of SATs or for the alumnus who has solved the latest problem in mathematics or science, are creating a mathe-

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matics culture. When children see the respect provided to academic excellence by a community they themselves begin to respect academic achievement. A society has to respect a human endeavor before children will begin to excel in it. Soccer was almost unknown to american schools only two decades ago, today almost in every suburban school, children engage in soccer matches. Parents are devoting enormous amount of time and encouragement to these children. In a sense, soccer was introduced to children and then they are being encouraged in it. I am sure in no time America will have a decent team to be able to compete at the international level. That shows if parents get interested in something, they promote it. That kind of zeal is necessary at the mathematics level, if we want our children to do well in mathematics and like doing it. Parents have to become party to it, today shools alone can not do it.

8. Homes and schools where children are encouraged to engage in free-flowing discussions are better prepared for analytical subjects such as mathematics and science than where the discussions and interactions between children and adults and even among children are rare.

Ways to develop mathematics culture in schools and home

In Home

1. Everyone who has tried to help a child in reading knows that one has to help the child acquire readiness for reading skill before actual reading by the child can take place. This holds true for mathematics also, with the exception that what is a prerequisite for one mathematics concept becomes a support skill for another mathematics concept. For example, classification is a prerequisite skill for subtraction and that in turn becomes a support skill for division and fractions. Therefore, in any support program for a child at home (or in school) whether the child is eight years old or fifteen, one needs to devote a good portion of one's time in helping the child acquire prerequisite and support skills. This applies to a parent who feels comfortable in teaching mathematics and wants to teach her child mathematics or a parent who would like to help her child in mathematics but does not feel competent in the content of mathematics and is uncomfortable in teaching mathematics. The parent in the second case may not like to (because of her own fear of mathematics) devote time on actual mathematics teaching but can still be able to help her child almost equally well if she devotes time in the development of these prerequisite and support skills. In our estimation, some of the important prerequisite and support skills are:

* classification
* matching and one-to-one correspondence
* following sequential directions
* spatial orientation/space organization
* estimation
* visualization
* pattern recognition and its extensions
* deductive/logical thinking
* inductive and intuitive thinking

The presence of these preskills in a child's repertoire facilitates mathematics learning. And in that sense these skills are "anchoring" skills: mathematics ideas get anchored on to these skills. When these skills are not present, a child learns a mathematical idea and two days later he may forget it because that idea was not anchored on to something. The development of these preskills takes place in an environment where the child has a lot of interaction with concrete, manipulative materials such as games and toys at home and concrete instructional materials and models in her school. These skills are best acquired in an informal, interactive environment. When children are playing with games and toys they are naturally engaged in learning many of these preskills. These games and toys have some basic characteristics:

1. These are games where some strategies are involved in being proficient in the game, i.e., with each encounter with the game or toy the child discovers something more about the game. And the game reflects the strategy being used. Such games are interesting to a novice and expert alike.

2. These games last on the average 15 to twenty minutes so that the child can see the end in a fairly short period of time (only when a child has more interest and is mature and is able to handle delayed gratification then only complex strategy games such as chess are meaningful to children. For some children, games such as chess sometimes become ends in themselves, which is fine, but then they no longer are serving the same purpose as we advocate--preparation for prerequisite skills for mathematics learning).

3. These games involve one or more of these preskills both directly or indirectly (that is why many of these manual dexterity computer games are not appropriate, but exploration games on computers are all right). For example, the commercially available game **Master Mind** is an excellent means for developing pattern recognition and deductive thinking in children and even adults.

During their play with these kinds of toys and games children learn and master these skills and eventually these skills are available to them when they need them in mathematics conceptualization.

Many parents ask me, "What can I do to help my child learn mathematics, particularly when I have had limited mathematical training and also little understanding especially if it is "New Math?" My answer to this question is that a parent should help her child with mathematics but only if you yourself feel comfortable with mathematics, otherwise one is likely to transfer some of the anxieties and fear of mathematics to one's child. To help a child in mathematics effectively requires that one understands mathematics concepts and also know how a child learns mathematics. If a parent helps her child in mathematics and the child does not respond positively the parent may begin to think that the child has something wrong with him/her. And as a result, the parent may begin to see the child in the light of the difficulty he/she is having and the parent's behavior may change towards him/her. Most importantly, a parent may not have the patience that one needs to work with a child who may be having some problems and difficulty in learning mathematics. A parent's behavior and reactions with the child may reflect expectations for the child negatively when one is working with one's child. Those expectations may not be right at that time. But there is a great deal a parent can do for her child. Firstly, one should create a mathematics culture in the home and secondly the parents can help their children a great deal when they help them acquire these preskills. As a matter of fact, a parent would have done half the work of a teacher by helping her child acquire these skills. If parents devote a lot of time with their children playing those games that will help inculcate these preskills then they would have helped their children to love mathematics and consequently children will find learning mathematics much easier. For a list of games that help develop these preskills and support skills see **Prerequisite and Support Skills For Mathematics Learning** (Math Notebook, Volume 2, Number 1, 1981).

2. Regular discussions are a good way to get children to talk about concepts while sharpening their thinking skills such as their ability to make inferences, to back up arguments with reasons and to make analogies, skills essential to learning and applying of mathematical skills. In an environment, where discussions are encouraged, children begin to ask questions not only of their classmates and siblings but from parents, and learn to evaluate the answers, draw conclusions and follow up with more questions both of converging and diverging type, which provides them facility in reasoning. Reasoning is the core of mathematics learning, and these discussions prepare children for reasoning, otherwise students become only procedurally oriented and too much of procedural, "recipe" learning eventually leads to boredom in mathematics. Mathematics culture is not just emphasizing formal mathematics learning in a formal setting but it also involves the emphasis on skills that relate to mathematics learning. Reasoning is one of those skills.

3. Parents are children's first teachers, but television is a close second. Both have fallen down on the job. The decline of the family involvement in learning and failure of television to serve children and education, particularly mathematics learning is widely known. But, how this can be remedied is ignored. Schools and parents need to make both the family and television partners in their efforts to promote the mathematics culture in homes and also via television both in schools and homes. As far as involvement of parents in the development of mathematics culture at home is concerned, schools must help parents to understand the elements of mathematics culture and mathematics learning by arranging lectures and workshops for parents. Schools should plan substantial activities for them and with them to encourage mathematics learning for both their children and for themselves. Parent/teacher organizations, at present, are not effectively used for this purpose. As far as media is concerned, schools need to use modern methods of promotion of their activity and their philosophy. They need to help media in understanding the learning process and clearly demonstrate to the media what programs are better for children by promoting those programs vigorously. Schools also need to help the media in translating, in simple terms, what good learning is and what are the affects of all learning on children's lives and in particular the effects of mathematics learning in children's future professional lives. It has to be a concerted effort.

In Schools

1. Teachers and school personnel need to show their commitment that they value mathematics learning. This is demonstrated by:

   * Rewarding children and their teachers by publicly recognizing them through multi-media.

   * Establishing healthy competitions throughout schools in mathematics achievement both at the formal and informal levels.

   * Bringing professionals (who use mathematics) to classrooms to expose children to them and their work.

2. There are specialized schools and institutions for every human endeavors where not only gifted and talented can benefit but also others who get interested in those areas can also benefit. Although every high school should provide the best possible mathematics environment, but at the same time, it is also good to establish special mathematics classes even in earlier grades for the very gifted and talented students staffed by teachers who are proficient and interested in mathematics. On the state level, there should be one or two special mathematics oriented high schools which will attract talented and mathematically precocious students and to be nurtured there. In these schools, all subjects will be taught but every subject will be taught by considering mathematics at its core. Soviet Russia, China, and Japan and some states in the United States have successfully utilized this concept.

3. Schools should constantly strive in training and retraining teachers in improving their understanding of mathematics. Schools should provide inservice in and exposure to their teachers in the latest information and proven research in mathematics learning and teaching methodologies. This should be done during and as well as after school hours so that the teachers realize that the school administration cares about the teaching and learning of mathematics and they also demonstrate it by this kind of commitment. Every corporation provides training and retraining to their staff on corporation's time, same type of commitment must be shown by our schools. During the last decades, schools have spent millions and millions of dollars on bringing all kinds of technologies in schools with the hope that the great amount of infusion of technology automatically will upgrade children's performances in mathematics and other related areas. It has made some impact, but a lasting effect will not take place if teacher competence in mathematics and their understanding of how children learn mathematics is not upgraded. Almost an equal amount of funds must be spent on teacher inservice for them to acquire competence in mathematics content and skills in teaching mathematics and also to understand and use the infusion of technology into schools. Only those teachers who have the competence in content and good understanding of children's learning styles, capabilities, and learning potential can effectively use the emerging technologies. Otherwise these technolo-

gies will overwhelm them and a few years from now these newly acquired technologies will have the same fate as other innovations of the past have ultimately faced.

4. Schools should give freedom for some experimentation and testing of new ideas in learning and teaching mathematics for those teachers who have shown high motivation and capabilities in mathematics learning and teaching. New approaches to teaching and learning can emerge not only from formal research from universities by researchers but also from teacher/researchers in schools themselves.

5. Inviting mathematicians and mathematics educators from the area universities, colleges and corporations to become involved in mathematics activities at the school level by:

--forming mathematics clubs
--allowing graduate students to teach and tutor in schools
--pairing school teachers with university and college mathematicians and researchers to conduct research and tryout strategies in the classroom
--inviting children to their work places in business and universities to show how mathematics is used there
--soliciting their help in teaching special topics to children showing the applications of mathematics in real life

In Communities

1. Area businessmen should get involved in schools. They can do it in three ways:

--exposing children to show how mathematics is used in their businesses.
--the business community, at present, recognizes and promotes all kinds of sport and athletic activities showing to children what actually matters. Business leaders need to change their attitudes about academic areas, they should put equal emphasis on academic and athletic areas. Whole communities can create and foster mathematics culture. Community leaders need to make conscious efforts to recognize outstanding achievement in mathematics at the school level. They should also sponsor projects, competitions, and interactions at local levels on the lines of what Westinghouse Science Talent Search does at the national level. Well organized, well attended, well publicized receptions should take place for the achievers in mathematics, returning teams form math olympiads, debating contests, science talent competitions, and other academic achievements and should be covered by the local TV and newspapers. These students should be awarded generous scholarships by area businesses and industry. The local shopping malls can display children's work and hold recognition sessions for outstanding performers in academics.
--businessmen should visit school classrooms and talk to children about mathematics and the ways it is used in real world and try to establish one to one relationships with them so that children can form identifications with achievers and mathematics users.

There has to be a deliberate effort to develop this mathematics culture in homes and schools so that children can see the importance of mathematics in this technologically complex society. Our failure in this will result in our facing a future where few will become experts in mathematics thereby creating a priestly class who will become the repository of mathematics knowledge. This situation has the germs of creating a class society--one with knowledge and expertize and another without. We are, to some extent, already seeing the signs of such a society. The spectre of such a society is the creation of a system which goes against all the principles on which this country was founded upon. It is possible to keep a fairly classless society, but that requires work on the part of those who are responsible for children, both in and outside of schools.

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